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INFLUENCE OF VISCOSITY ON THE THERMAL INSTABILITY

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Summary: A numerical analysis of the viscous dissipation influence on the thermal instability of a medium with thermal conduction and heating&cooling function without any gravitation and magnetic field is presented. For the condensation and wave modes the effects of viscosity on the dependence $k(\omega)$ and scale of maximum instability are distinguished. It is found that these effects are significantly larger for the wave mode.

T. Angelov: UTICAJ VISKOZNOSTI NA TOPLOTNU NESTABILNOST – Daje se numerička analiza uticaja viskozne disipacije na toplotnu nestabilnost sredine sa toplotnim provodjenjem i funkcijom zagrevanja i hladjenja, bez gravitacione sile i magnetnog polja. Za kondenzacione i talasne mode, izdvajaju se efekti viskoznosti na zavisnost $k(\omega)$ i na skalu maksimuma nestabilnosti. Pokazuje se da su ovi efekti znatno veći za talasnu modu.

1. INTRODUCTION

In an earlier paper (Angelov, 1988) the thermal instability of a viscous medium without any gravitation and magnetic field was considered. For small perturbations of the type $\exp(\omega t + i\vec{k}\cdot\vec{r})$ and for $\nabla \times \vec{V} = 0$ (a weak viscosity influence on the length λ) the following characteristic equation was obtained

$$z^3 - Az^2 + B_i z - D_i = 0 \tag{1}$$

where

$$z = \frac{\omega}{kc}, \; A = -\left(\frac{k_T}{k} + \frac{k}{k_\kappa} + \frac{k}{k_\nu}\right). \tag{2}$$

For purely real solutions of (1) – the condensation instability mode ($i = 1$),

$$B_1 = 1 + \frac{k}{k_\nu}\left(\frac{k}{k_\kappa} + \frac{k_T}{k}\right), \; D_1 = \frac{1}{\gamma}\left(A + \frac{k}{k_\nu} + \frac{k_\varrho}{k}\right) \tag{3a}$$

and for $\mathrm{Re}(z)$ of its conjugate-complex solutions – the wave instability mode ($i = 2$),

$$B_2 = \frac{1}{4}(A^2 + B_1), \; D_2 = \frac{1}{8}(AB_1 - D_1). \tag{3b}$$

In (2), (3a), (3b) the wavenumbers are introduced

$$k_T = K\mathcal{L}_T, \; k_\varrho = K\frac{\varrho}{T}\mathcal{L}_\varrho, \; k_\kappa = \frac{1}{K}\frac{\varrho}{\kappa}, \; k_\nu = \frac{\varrho c}{\eta_1 + \frac{4}{3}\eta_2}, \tag{4}$$

with $K = (\gamma-1)\mu/Rc$, where R, μ, ϱ, T are the gas constant, the mean molecular weight, the density and the temperature, respectively, $\gamma = c_p/c_v$, $c = (\gamma p/\varrho)^{1/2}$ – the adiabatic speed of sound, κ and η_1, η_2 – thermal conduction coefficient and both viscosity coefficients; $\mathcal{L}_\varrho, \mathcal{L}_T$ are the partial derivatives of the function $\mathcal{L}(\varrho, T)$ in ϱ and T. The latter function is defined as energy losses minus energy gains, per gram of material per second. All quantities from (4) are calculated in the basic state of the medium which is homogeneous, being in mechanical and thermal equilibrium.

For $k \in \mathrm{Re}$ the medium is unstable in the domain $D_i > 0$, whereas the perturbations with $k > k_{c,i}$ are stabilized by the dissipation processes. If k_ϱ and k_T are small,

$$k_{c,1} = \{k_\kappa(k_\varrho - k_T)\}^{\frac{1}{2}}, \; k_{c,2} = \left\{-k_\kappa \frac{k_T + \frac{1}{\gamma-1}k_\varrho}{1 + \frac{\gamma}{\gamma-1}\cdot\frac{k_\kappa}{k_\nu}}\right\}^{\frac{1}{2}} \tag{5}$$

for the condensation and wave modes, respectively.

In this paper a more detailed analysis of the viscous dissipation influence on the dependence $k(\omega)$ within an unstable region (Sect. 2), as well as on the maximum instability (Sect. 3), is presented.

2. CHARACTER OF PERTURBATION STABILIZATION

Let the dependence $k(\omega)$ in a given medium be considered. By introducing dimensionless quantities

$$u = (k/k_\varrho)^2, \ \alpha = k_T/k_\varrho, \ \beta_\kappa = k_\varrho/k_\kappa, \ \beta_\nu = k_\varrho/k_\nu, \tag{6}$$

the characteristic equation (1) is transformed into the form

$$a_i u^3 + b_i u^2 + c_i u + d_i = 0. \tag{7}$$

The coefficients in (7) are

$$\begin{aligned} a_1 &= 0, \ b_1 = \beta_\kappa/\gamma, \\ c_1 &= (\beta_\kappa + \beta_\nu)x^2 + (1 + \alpha\beta_\nu)x + (\alpha - 1)/\gamma, \\ d_1 &= x^2(x + \alpha) \end{aligned} \tag{8a}$$

for the condensation mode ($i = 1$), where $x = \omega/ck_\varrho$ and

$$\begin{aligned} a_2 &= (\beta_\kappa + \beta_\nu)\beta_\kappa\beta_\nu, \\ b_2 &= 2[(\beta_\kappa + \beta_\nu)^2 + \beta_\kappa\beta_\nu]\xi + \frac{\gamma - 1}{\gamma}\beta_\kappa + [1 + \alpha(2\beta_\kappa + \beta_\nu)]\beta_\nu, \\ c_2 &= 8(\beta_\kappa + \beta_\nu)\xi^2 + 2[1 + \alpha(2\beta_\kappa + 3\beta_\nu)]\xi + \alpha^2\beta_\nu + \frac{1 + (\gamma - 1)\alpha}{\gamma}, \\ d_2 &= 2\xi(2\xi + \alpha)^2, \end{aligned} \tag{8b}$$

for the wave mode ($i = 2$), where $\xi = \mathrm{Re}(\omega/ck_\varrho)$. Note that quadratic part of (7) for a non-viscous medium ($\beta_\nu = 0$) was analysed by Field (1965).

For a dissipationless medium ($\beta_\kappa = \beta_\nu = 0$) the solution of (7) is $u = -d_i/c_i$ and $0 \le k \le \infty$ for

$$0 \le x \le \frac{1 - \alpha}{\gamma}(\alpha < 1), \ \text{i.e.} \ 0 \le \xi \le \frac{(1 - \gamma)\alpha - 1}{2\gamma}\left(\alpha < -\frac{1}{\gamma - 1}\right), \tag{9}$$

for $i = 1$, $i = 2$, respectively. In a dissipative medium ($\beta_\kappa, \beta_\nu \ne 0$) the unstable wavenumbers domain is $0 \le k \le k_{c,i}$ where

$$k_{c,1} = k_\varrho\left(\frac{1 - \alpha}{\beta_\kappa}\right)^{\frac{1}{2}}, \ k_{c,2} = k_\varrho\left\{-\frac{\alpha + \frac{1}{\gamma - 1}}{\beta_\kappa + \frac{\gamma}{\gamma - 1}\beta_\nu}\right\}^{\frac{1}{2}} \tag{10}$$

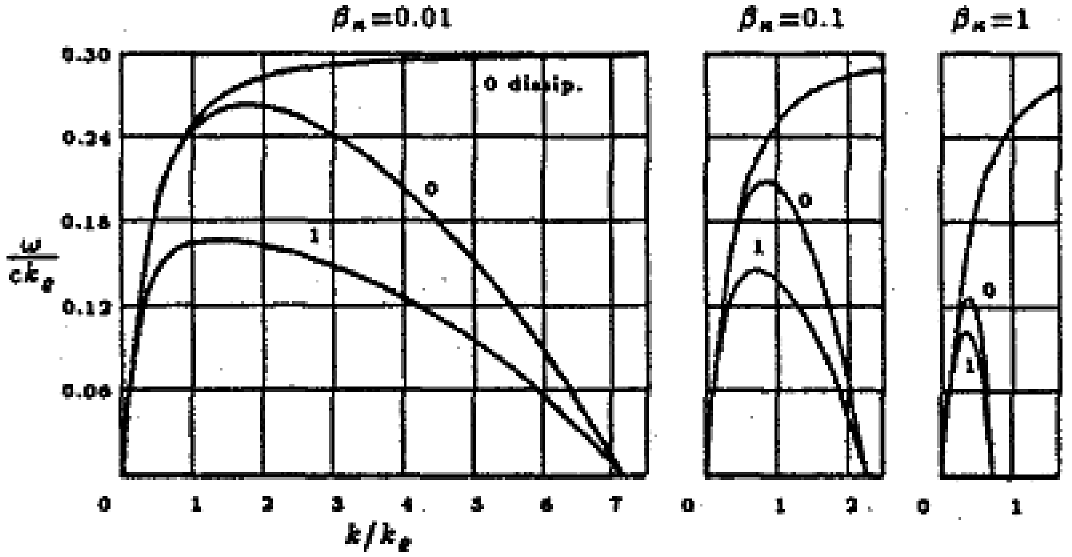

Fig. 1 – Viskosity influence on unstable domain *of condensation mode* ($\gamma = 5/3,\ \alpha = 1/2$) : $\beta_\nu = 0, 1$ for each $\beta_\kappa = 0.01, 0.1, 1$.

The expressions yielding $k_{c,i}$ are obtained from (7) for $x = 0$, i.e. $\xi = 0$ or directly from (5) with the aid of (6). The value $k \neq 0$ for the marginal instability of the wave mode is equal to $k_{c,2}$ from (10) only in the linear approximation of the real root $u^{1/2}$ of the equation

$$a_2u^2 + b_2u + c_2 = 0,$$

when β_κ and β_ν are small quantities. The roots $u^{1/2}(x)$ and $u^{1/2}(\xi)$ of equ. (7), which is quadratic in u for the condensation mode and cubic in u for the wave mode, are presented in Figs. 1–2 for given $\gamma, \alpha, \beta_\kappa$ and β_ν.

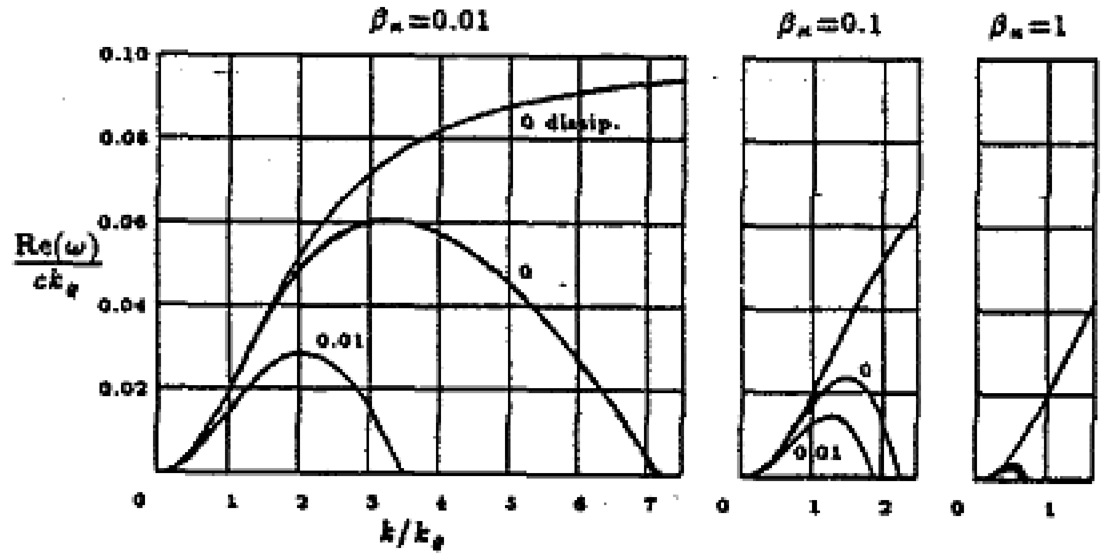

Fig. 2 – Viscosty influence on unstable wave-mode domain ($\gamma = 5/3,\ \alpha = -2$) : $\beta_\nu = 0, 0.01$ for each $\beta_\kappa = 0.01, 0.1, 1$.

3. SCALE OF THE MAXIMUM INSTABILITY

It is seen from Figs. 1–2 that a maximum of instability is present for $x = x_m$, i.e. $\xi = \xi_m$. These values, for $\beta_\kappa, \beta_\nu \neq 0$, are obtained from the condition that equ. (7) has one double root. For the condensation mode, x_m and $u_{m,1} = u(x_m)$ are determined from

$$c_1^2 - 4b_1d_1 = 0$$
$$2b_1u_{m,1} + c_1 = 0. \tag{11}$$

For the wave mode ξ_m and $u_{m,2} = u(\xi_m)$ are solutions of the system

$$4p^3 + 27q^2 = 0, \quad u_{m,2} - (q/2)^{\frac{1}{3}} + b_2/3a_2 = 0$$

(p and q are coefficients of the normal form of the cubic (7)), i.e. they are obtainable from

$$b_2u_{m,2}^2 + 2c_2u_{m,2} + 3d_2 = 0$$
$$3a_2u_{m,2}^2 + 2b_2u_{m,2} + c_2 = 0. \tag{12}$$

By solving (11), i.e. (12), with coefficients dependent of x_m, i.e. ξ_m, one obtains the equations

$$P_i(y_m) \equiv \sum_{k=0}^{n}\sum_{j} E_{i,jk}(\gamma,\alpha,\beta)\beta_\kappa^j y_m^k = 0, \tag{13}$$

$$Q_i(u_m) \equiv \sum_{k=0}^{n}\sum_{j} F_{i,jk}(\gamma,\alpha,\beta)\beta_\kappa^j u_{m,i}^k = 0, \tag{14}$$

for unkown quantities y_m and $u_{m,i}$, where (i, n, y_m) is $(1, 4, x_m)$ for the condensation mode, i.e. $(2, 6, \xi_m)$ for the wave mode. The coefficients in (13) and (14) depend on the medium parameters ($\gamma, \alpha, \beta_\kappa$ and $\beta = \beta_\nu/\beta_\kappa$) only, but it is unnecessary to give their expressions here (some of them vanish, but some of them are very clumsy). In any case the form of the coefficients (as polynomials in β_κ) makes possible examinations of the solution behaviour within the domain of small β_κ (large $u_{m,i}$), which is the most frequent case in praxis. Let the approximative dependence in this domain be

$$u_{m,i} = g_i\beta_\kappa^{-l_i}, \ i = 1, 2 \tag{15}$$

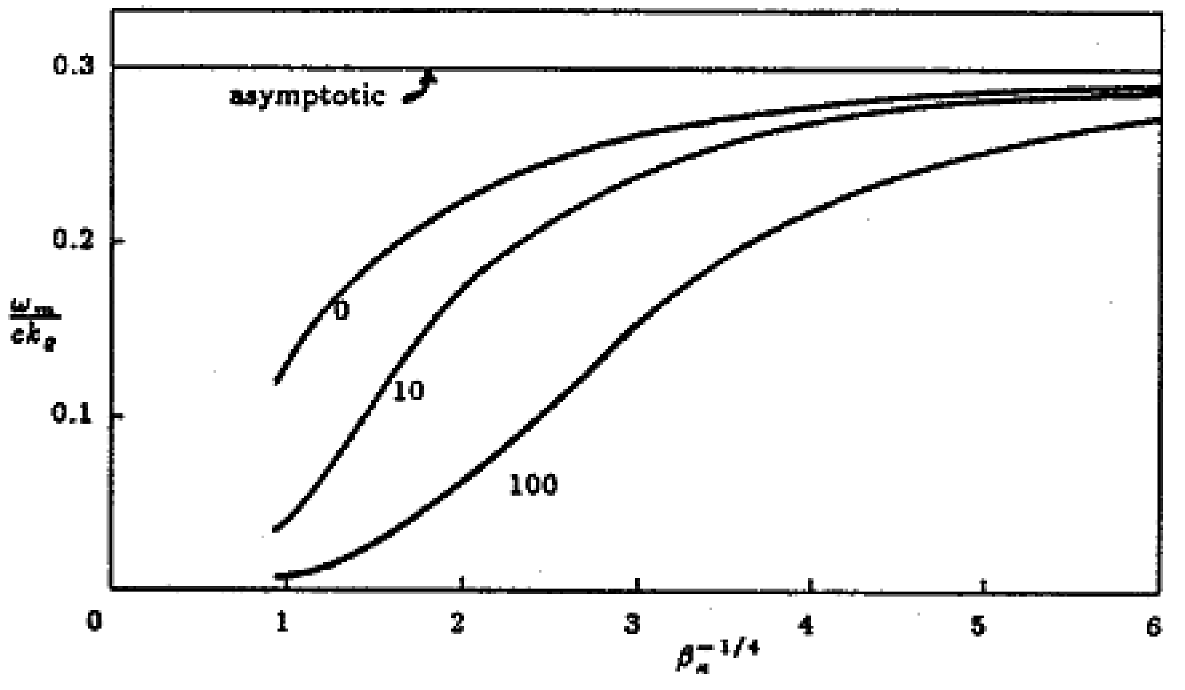

Fig. 3a – Normalized ω values at maximum instability for the condensaton mode ($\gamma = 5/3$,$\alpha = 1/2$), from (13), for $\beta = 0, 10, 100$. Asymtotic x_m value from (9).

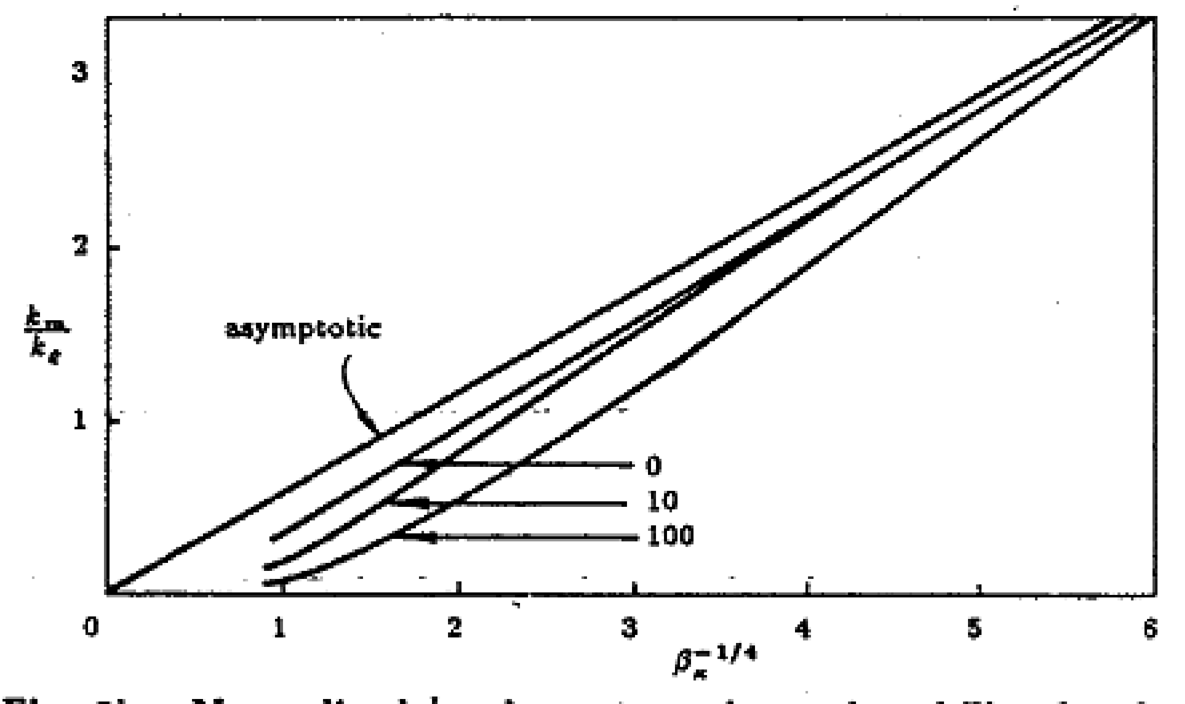

Fig. 3b – Normalized k values at maximum instability for the condensation mode ($\gamma = 5/3, \alpha = 1/2$), from (14), for $\beta = 0, 10, 100$. Asymtotic dependence from (19).

where g_i and l_i are real, positive, constants (for $\beta_\kappa \ll 1$, y_m strives to the limiting values x_m, i.e. ξ_m from (9) for short wavelengths). The general term in (14) is then $F_{i,jk} g_i^k \beta_\kappa^{s_i}$, with $s_i = j - kl_i$, and the dominant terms of the polynomial $Q_i(\beta_\kappa), \beta_\kappa < 1$, are those with $s_i = s_{i,\min}$. For $s_i = 0$, the dominant part of (14) is

$$F_{i,00} + g_i^2 F_{i,12} = 0 \tag{16}$$

where

$$F_{1,00} = -\left(\frac{1-\alpha}{\gamma}\right)^2 \frac{1+(\gamma-1)\alpha}{\gamma}, \quad F_{1,12} = \frac{1}{\gamma}; \tag{17a}$$

$$F_{2,00} = -\left(\frac{1-\alpha}{\gamma}\right)^2 \frac{(1-\gamma)\alpha-1}{\gamma}, \quad F_{2,12} = \frac{\gamma-1}{\gamma} + \beta \tag{17b}$$

for the condensation mode and wave mode, respectively, and

$$l_i = j/k = \frac{1}{2}, \ i = 1,2 \tag{18}$$

for both of them. From the solution of (15) with the aid of (18) one obtains

$$k_{m,i} = k_\varrho G_i \beta_\kappa^{-1/4}, \ i = 1,2 \tag{19}$$

where $G_i = g_i^{1/2}$. From (16) and (17a), (17b),

$$G_1(\gamma,\alpha) = \left(\frac{1-\alpha}{\gamma}\right)^{1/2} \{1+(\gamma-1)\alpha\}^{1/4}, \tag{20a}$$

$$G_2(\gamma,\alpha,\beta) = \left(\frac{1-\alpha}{\gamma}\right)^{1/2} \left\{\frac{(1-\gamma)\alpha-1}{\gamma\left(\frac{\gamma-1}{\gamma}+\beta\right)}\right\}^{1/4}. \tag{20b}$$

The effect of viscosity on normalized ω and k is presented in Figs. 3a,b and in 4a,b for the condensation and wave mode, respectively. It is seen that at the maximum instability both ω and k decrease when the total dissipation increases (in accordance with the way of reduction of unstable domains in Figs. 1–2). Depending on β_κ (β is a free parameter), both $\omega_{m,i}$ and $k_{m,i}$ are variable within the domain $\beta_\kappa < \beta_{\kappa,i}$, where $\beta_{\kappa,i}$ is determined by the condition $k_{m,i} = 0$ i.e. $\sum_j F_{i,j0}\beta_{\kappa,i}^j = 0$ from (14). For the condensation mode, $\beta_{\kappa,1} = [1+(\gamma-1)\alpha]/\gamma\alpha^2 > 1$ (for $\gamma = 5/3$, $\alpha = 1/2$, $\beta_{\kappa,1} = 3.2$) i.e., $\beta_\kappa = 1$ is the real limitation for the thermal conductivity influence. An analogous consideration for the wave mode yields

$$\beta_{\kappa,2} = \frac{(1-\gamma)\alpha-1}{\gamma\alpha^2\beta} \tag{21}$$

(for $\gamma = 5/3$, $\alpha = -2$: $\beta_{\kappa,2} = 0.05/\beta$).

By using (10) and (20a), (20b), $k_{m,i}$ from (19) is

$$k_{m,1} = \left\{\left(\frac{1-\alpha}{\gamma}\right)^2 + \alpha\frac{1-\alpha}{\gamma}\right\}^{1/4} (k_\varrho k_{c,1})^{1/2}, \tag{22a}$$

$$k_{m,2} = \left(\frac{1-\alpha}{\gamma}\right)^{1/2} (k_\varrho k_{c,2})^{1/2}. \tag{22b}$$

For $\beta \ll 1$, G_i and $k_{m,i}$ for the wave mode can be written as

$$G_2 = G_2^* f(\beta), \; k_{m,2} = k_{m,2}^* f(\beta); \; f(\beta) \approx 1 - \frac{1}{4} \cdot \frac{\gamma}{\gamma - 1}\beta, \tag{23}$$

where G_2^* is G_2 from (20b) for $\beta = 0$ and $k_{m,2}^*$ is $k_{m,2}$ from (22b) with $k_{c,2}$ from (10), for $\beta = 0$. In any case for the purpose of approximative calculating it remains $k_{m,i} \approx (k_\varrho k_{c,i})^{1/2}$, for both instability modes (Field, 1965) — the correction applied to $k_{c,2}$ with respect to a non-viscous medium, is small.

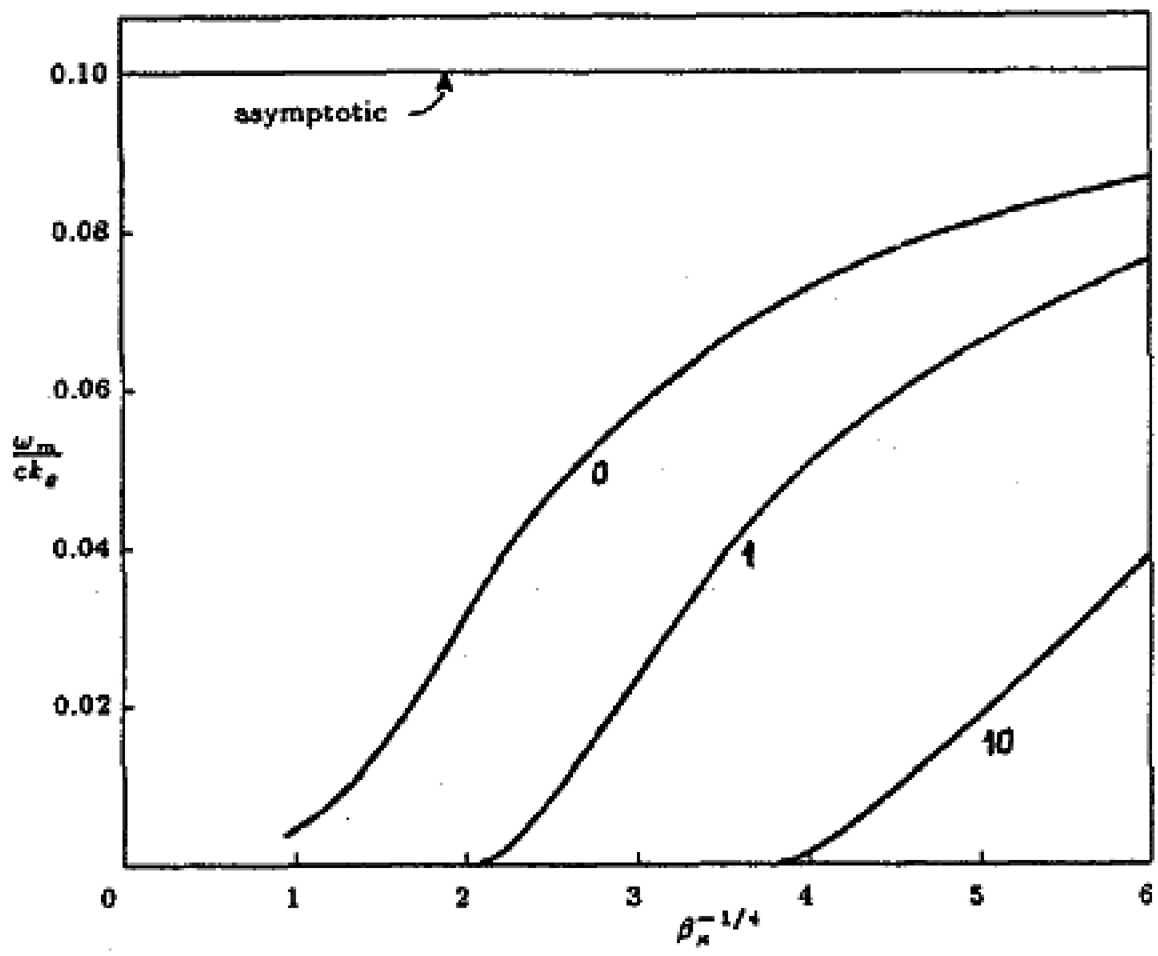

Fig. 4a – Normalized ω values at maximum instability for the wave mode ($\gamma = 5/3, \alpha = -2$), from (13), for $\beta = 0, 1, 10$.
Asymtotic value from (9).

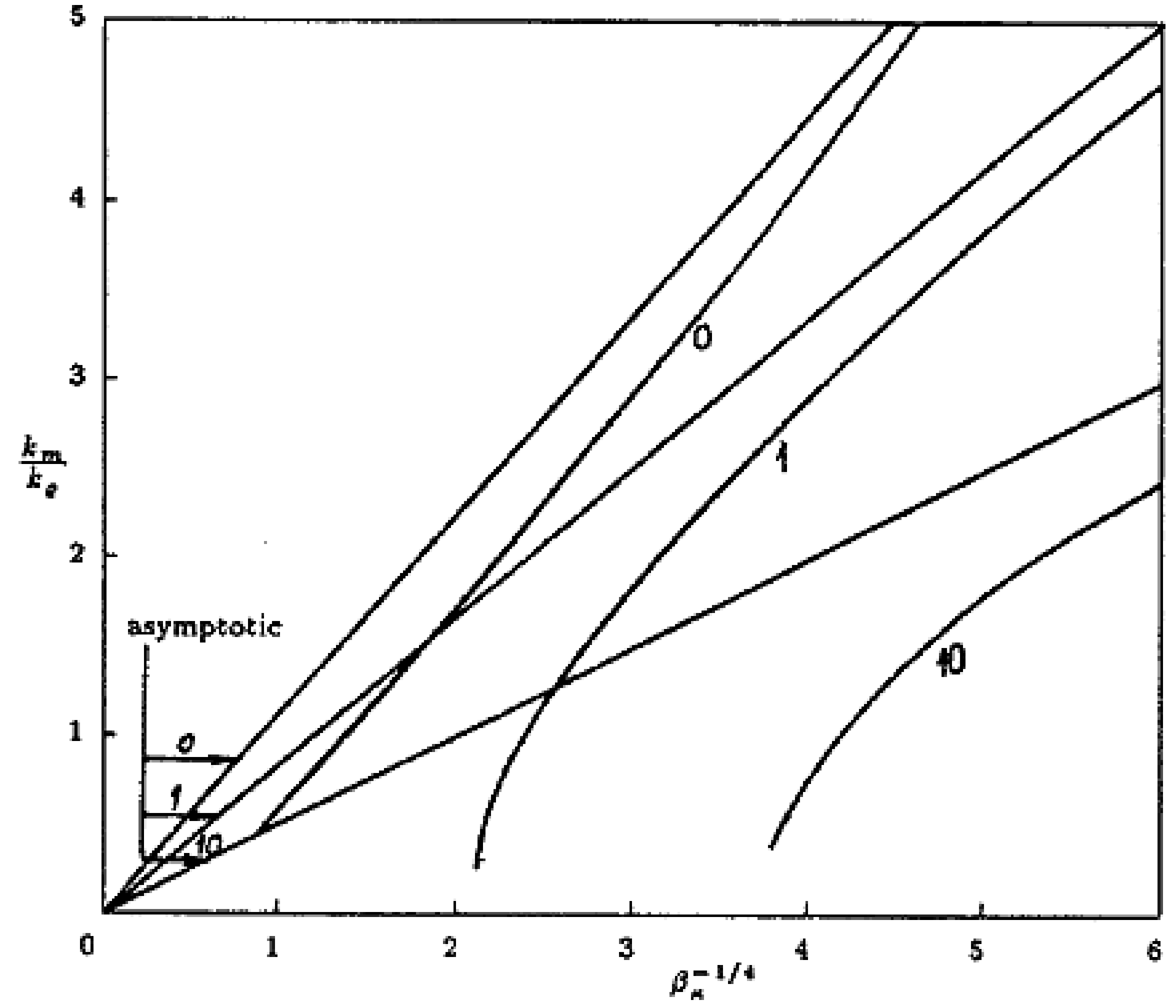

Fig. 4b - Normalized k values at maximum instability for the wave mode ($\gamma = 5/3, \alpha = -2$), from (14), for $\beta = 0, 1, 10$. Asymtotic dependences from (19).

4. CONCLUSION

The subject of the present paper is a small viscous dissipation influence on the thermal instability of a medium. In such a model a redistribution of perturbations due to viscosity takes place, but the marginal instability of the condensation mode is unaffected (the limiting wavelength is determined by thermal conduction effect only). The influence of the viscous dissipation on the wave mode is stronger – it reduces the marginal instability domain and stabilizes the perturbations completely already at $\beta_\nu = [(1-\gamma)\alpha - 1]/\gamma\alpha^2$ (in the numerical example treated here: $\gamma = 5/3, \alpha = -2$, the wave instability mode exists solely for $\beta_\nu < 0.05$). In the case of small thermal conductivity influences ($\beta_\kappa < 0.01$) the maximum-instability scale of the condensation mode is practically unaffected for $\beta_\nu \leq \beta_\kappa$, but becomes significantly reduced for the wave mode (up to 50% compared to a non-viscous medium). The relative deviation of the asymtotic solution for k_m from the exact one within the domain $\beta_\kappa < 0.01, \beta_\nu \le \beta_\kappa$, is less than 10% for the condensation mode and less than 30% in the wave-mode case.

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