General Relativity and Gravitation, Vol. 22, No. 7, 1990

Conformally Equivalent Metrics in Bimetric General Relativity

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Received October 11, 1988

Space-times conformal to physical space-time are considered, assuming the nongravitational energy is conservative after the conformal transformation. Necessary and sufficient conditions satisfied by the conformal factor are found for a given type of transformation of the energy tensor. The weak gravitational field is defined and the coordinate conditions for the existence of conformal factors in such a field are obtained.

1. INTRODUCTION

Rosen's bimetric general relativity represents a variant of alternative theories of gravitation. It is in fact a modification of classical general relativity, as the space-time is assumed to be of constant curvature in the absence of any form of energy. That assumption is expressed by an additional term at the left hand side of the gravitational field equations [1,2]. The consequences of that term are important. The law of energy conservation is no longer the automatic consequence of Bianchi's identity. It requires an additional condition which ties together the metric tensor of the background space-time (the universe free of any form of energy) with the metric tensor of the physical space-time, and represents a generalization of the De Donder condition of classical relativity.

Among the essential consequences of bimetric general relativity is a background coordinate system, associated with the universe. Its funda-

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0001-7701/90/0700-0721$06.00/0

mental effects in the solar system should not be discernably different from those of classical general relativity. Its cosmological solutions have models without a "Big Bang" [1,2]. The structure of ordinary stars agrees with the predictions of classical relativity, with the difference that a configuration in hydrostatic equilibrium exists inside the Schwarzschild sphere in the case of collapsed stars [3]. In our mind these consequences of bimetric general relativity, with the possible exception of the background coordinate system, recommend it as a serious alternative to classical relativity.

After a brief survey of the general results of the theory in Section 2, we consider in Section 3 the question of the conformally related metrics satisfying the gravitational field equations. In Rosen's bimetric gravitation theory (which is not to be confused with bimetric general relativity) conformally equivalent metrics, satisfying the gravitational equations, always exist under simple conditions [4,5]. In bimetric general relativity the situation is quite different. First, the nongravitational energy tensor T_{μ}^{ν} cannot be conserved in both metrics, unless they are mutually related by a constant conformal factor. We have chosen to investigate the case when only the conformally transformed metric is conservative, the conservation of nongravitational energy not being, as mentioned, the consequence of an identity. We have considered the conformally equivalent metric to be as interesting as the initial one. The consequences of that choice are, first, a basic relation between the physical metric coefficients and the conformal factor, then a general relation between the divergences of the initial and the transformed nongravitational energy tensors. Restricting our investigation to homogeneously transformed energy tensors, we obtain the necessary and sufficient condition satisfied by the conformal factor in the case of energy conservation and a condition on the physical metric resulting therefrom. Finally, a few cases of homogeneous transformations are considered.

In Section 4 previous results are applied to weak fields, defined in a way analogous to that of classical relativity. It appears that a necessary condition for the existence of a conformal factor can be fulfilled under infinitesimal coordinate transformations suited to weak fields. Finally, homogeneously transformed energy tensors in a weak gravitational field are discussed.

2. GENERAL RELATIONS

In bimetric general relativity one considers two line elements

$$\epsilon\, d\sigma^2 = \gamma_{\alpha\beta}\, dx^\alpha\, dx^\beta \tag{1}$$

and

$$\epsilon\, ds^2 = g_{\alpha\beta} dx^\alpha\, dx^\beta . \tag{2}$$

Element (1) corresponds to a background space-time of constant curvature, to which the physical metric reduces in the absence of any kind of energy. The element (2) defines the physical metric. The assumptions concerning the differentiability of the physical metric are as usual; $g_{\alpha\beta}$ is differentiable C^1, C^3 piecewise.

The gravitational field equations read

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} - \frac{3}{a^2}(\gamma_{\mu\nu} - \frac{1}{2} g^{\alpha\underline{\alpha}} g_{\mu\nu}) = -8\pi T_{\mu\nu}. \tag{3}$$

One sees that (3) differs from the Einstein field equations by an additional term on the left hand side. Hereafter, an underlined index means that it is lowered (resp. raised) with the help of $\gamma_{\mu\nu}$. We shall denote by a bar (|) a covariant derivative with respect to the metric $\gamma_{\mu\nu}$; by a semicolon (;) a covariant with respect to the metric $g_{\mu\nu}$; by a comma (,) a partial derivative.

A basic bimetric formula reads [2]

$$\{^{\lambda}_{\mu\nu}\} = \Delta^\lambda_{\mu\nu} + \Gamma^\lambda_{\mu\nu} \tag{4}$$

where the Christoffel symbol on the left hand side corresponds to $g_{\mu\nu}$, $\Gamma^\lambda_{\mu\nu}$ corresponds to $\gamma_{\mu\nu}$ and $\Delta^\lambda_{\mu\nu}$ has the form

$$\Delta^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\rho}(g_{\mu\rho|\nu} + g_{\rho\nu|\mu} - g_{\mu\nu|\rho}). \tag{5}$$

On account of (4) and (5) we have

$$R^\lambda{}_{\mu\nu\rho} = K^\lambda{}_{\mu\nu\rho} + P^\lambda{}_{\mu\nu\rho}. \tag{6}$$

Here $R^\lambda{}_{\mu\nu\rho}$ and $P^\lambda{}_{\mu\nu\rho}$ are respectively the curvature tensors of the metrics $g_{\alpha\beta}$ and $\gamma_{\alpha\beta}$, so that

$$P^\lambda{}_{\mu\nu\rho} = \frac{1}{a^2}(\delta^\lambda_\rho \gamma_{\mu\nu} - \delta^\lambda_\nu \gamma_{\mu\rho}) \qquad (a = \text{const.}) \tag{7}$$

By (5) $K^\lambda{}_{\mu\nu\rho}$ is given by

$$K^\lambda{}_{\mu\nu\rho} = \Delta^\lambda_{\mu\rho|\nu} - \Delta^\lambda_{\mu\nu|\rho} + \Delta^\alpha_{\mu\rho}\Delta^\lambda_{\alpha\nu} - \Delta^\alpha_{\mu\nu}\Delta^\lambda_{\alpha\rho}. \tag{8}$$

This tensor differs from $R^{\lambda}{}_{\mu\nu\rho}$ in that the partial derivatives are replaced by γ-derivatives. Equation (6) holds for arbitrary curvature tensors instead of $P^{\lambda}{}_{\mu\nu\rho}$ [1,6].

Let us return to the field equations (3). From (4)–(8), equations (3) can be written in the form

$$H_{\mu\nu} \equiv K_{\mu\nu} - \frac{1}{2} K g_{\mu\nu} = -8\pi T_{\mu\nu}. \tag{9}$$

The mixed form of the relations (6) allows the curvature tensors, which depend on different metric tensors, to be simultaneously contracted with respect to one index. The second contraction, necessary for obtaining K, is done with the help of the physical metric tensor $g^{\mu\nu}$.

The conservation of nongravitational energy reads, as usual [1,2]

$$T^{\nu}_{\mu;\nu} = 0. \tag{10}$$

Since the left hand side of the gravitational equations (3) has additional terms, the covariant divergence does not automatically vanish. So eq. (10) implies one condition more, that is

$$[(g/\gamma)^{1/2} g^{\underline{\mu\nu}}]_{|\nu} = 0, \tag{11}$$

g and γ denoting, as usual, the respective determinants of the metrics. Equation (11) represents a generalization of the De Donder condition of classical relativity.

3. CONFORMAL TRANSFORMATIONS

As we already mentioned, in bimetric gravitation theory metrics, conformally equivalent to a given metric, always exist under simple conditions. From the basic system of equations (3), or (9), one should expect bimetric general relativity to have more similarities with classical general relativity than with bimetric gravitation theory. As we shall see, there are conditions which notably reduce the possibility of the existence of solutions which are both conformally equivalent to given ones and conservative.

If we perform a conformal transformation of the metric

$$\bar{g}_{\alpha\beta} = e^{2\phi} g_{\alpha\beta}, \tag{12}$$

the left hand side of the gravitational equations will obviously be modified in the same way as in classical relativity, and we shall have (Ref. 7, p. 317)

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + 2\phi_{;\mu\nu} - 2\phi_{,\mu}\phi_{,\nu} - 2g_{\mu\nu} g^{\sigma\tau}(\phi_{;\sigma\tau} + \frac{1}{2}\phi_{,\sigma}\phi_{,\tau})$$
$$= \frac{3}{a^2}(\gamma_{\mu\nu} - \frac{1}{2} g^{\sigma\underline{\sigma}} g_{\mu\nu}) - 8\pi \bar{T}_{\mu\nu} \tag{13}$$

where $\tilde{T}_{\mu\nu}$ is the conformally transformed nongravitational energy tensor. Assuming both $T_{\mu\nu}$ and $\tilde{T}_{\mu\nu}$ conservative in their respective metrics, we shall obtain from (10), or equivalently from (11), formulated in both metrics, that ϕ must be a constant. Two conformally equivalent metrics, satisfying (3) and (13) respectively, can exist and be mutually related by a nonconstant factor. But only one of them, by the preceding, allows the conservation of nongravitational energy.

We shall assume, in what follows, $g_{\mu\nu}$ and $\tilde{g}_{\mu\nu}$ satisfying respectively (3) and (13), and $\tilde{T}_{\mu\nu}$ conservative. Let us first write (13) in a form analogous to (3)

$$\tilde{K}_{\mu\nu} - \frac{1}{2}\tilde{K}\tilde{g}_{\mu\nu} = -8\pi\tilde{T}_{\mu\nu}. \tag{14}$$

The condition of conservation of $\tilde{T}_{\mu}^{\nu}$ in the metric $\tilde{g}_{\mu\nu}$ is equivalent to

$$[(\tilde{g}/\gamma)^{1/2}\tilde{g}^{\mu\nu}]_{|\nu} = 0 \tag{15}$$

just as (10) and (11) are mutually equivalent in the metric $g_{\mu\nu}$. Equation (15), explicitly written with the help of (12), yields

$$\phi_{,\epsilon} = \frac{1}{2}(g^{\alpha\beta}g_{\alpha\epsilon|\beta} - \lambda_{,\epsilon}) \tag{16}$$

where

$$\lambda_{,\epsilon} = \frac{1}{2g/\gamma}(g/\gamma)_{,\epsilon} = \frac{1}{2}g^{\alpha\beta}g_{\alpha\beta|\epsilon}. \tag{17}$$

By (16) the conservation of $\tilde{T}_{\mu}^{\nu}$ implies that $g^{\alpha\beta}g_{\alpha\epsilon|\beta}$ is a gradient.

It is obvious, by (16) and (17), that a metric conformal to the background metric allows only ϕ =const. The background metric, being of constant curvature, is itself conformally flat. The result is that any conservative conformally flat physical metric reduces, through a constant factor, to the background metric.

The necessary and sufficient condition for the right hand side of (16) to be a gradient reads, in terms of γ-derivatives,

$$g^{\lambda\nu}(g_{\epsilon\lambda|\nu\mu} - g_{\mu\lambda|\nu\epsilon}) + g^{\lambda\nu}g^{\rho\sigma}(g_{\lambda\sigma|\epsilon}g_{\rho\mu|\nu} - g_{\lambda\sigma|\mu}g_{\rho\epsilon|\nu}) = 0. \tag{18}$$

A physical metric, which allows a conformally equivalent metric in which nongravitational energy is conserved, does identically satisfy (18).

Consider now the energy tensor on the right hand side of (14). The condition of conservation of $\tilde{T}_{\mu}^{\nu}$, after the substitution of (12) in the respective divergence, reads

$$\tilde{T}_{\mu*\mu}^{\nu} = \tilde{T}_{\mu;\nu}^{\nu} + 4\tilde{T}_{\mu}^{\nu}\phi_{,\nu} - \tilde{T}\phi_{,\mu} = 0 \tag{19}$$

where the asterisk (*) denotes covariant differentiation with respect to $\bar{g}_{\mu\nu}$.

By (3), after some calculus, one obtains for the covariant divergence of T^{ν}_{μ}

$$T^{\nu}_{\mu;\nu} = \frac{3}{8\pi a^2 (g/\gamma)^{1/2}} [(g/\gamma)^{1/2} g^{\underline{\mu\nu}}]_{|\nu}, \tag{20}$$

From this fundamental formula of bimetric relativity result the mutually equivalent relations (10) and (11). By (15) and (16) the preceding formula can be written as

$$-\frac{4}{3}\pi a^2 g_{\epsilon\underline{\mu}} T^{\nu}_{\mu;\nu} = \phi_{,\epsilon} \tag{21}$$

so that (19) takes the form

$$\tilde{T}^{\nu}_{\mu;\nu} - \frac{16}{3}\pi a^2 g_{\nu\underline{\lambda}} (\tilde{T}^{\nu}_{\mu} - \frac{1}{4}\delta^{\nu}_{\mu}\tilde{T}) T^{\rho}_{\lambda;\rho} = 0. \tag{22}$$

In the case that $g_{\mu\nu}$ allows a conformally equivalent metric $\bar{g}_{\mu\nu}$, which is a solution of the gravitational equations, assuming that nongravitational energy is conserved, the covariant divergence of T^{ν}_{μ} and $\tilde{T}^{\nu}_{\mu}$ in the metric $g_{\mu\nu}$ are mutually related by the linear expression (22).

(a) Let us consider homogeneously transformed energy tensors

$$\tilde{T}^{\nu}_{\mu} = e^{n\phi} T^{\nu}_{\mu} \Rightarrow \tilde{T}_{\mu\nu} = e^{(n+2)\phi} T_{\mu\nu}. \tag{23}$$

Such conformal transformations of nongravitational energy tensors are physically simple and consistent. In the case of a pure electromagnetic field they are (with $n = -4$) the only ones justified [4,5].

Let us substitute $\tilde{T}_{\mu\nu}$ from (23) in (13), then subtract (3) from it; we shall have

$$2\phi_{;\mu\nu} - 2\phi_{,\mu}\phi_{,\nu} - 2g_{\mu\nu}g^{\sigma\tau}(\phi_{;\sigma\tau} + \tfrac{1}{2}\phi_{,\sigma}\phi_{,\tau}) = 8\pi[e^{(n+2)\phi} - 1]T_{\mu\nu}. \tag{24}$$

In deriving the above formula we assumed $e^{(n+2)\phi} \neq 1$. By (23), (19) takes the form

$$T^{\nu}_{\mu;\nu} + (n+4)T^{\nu}_{\mu}\phi_{,\nu} - T\phi_{,\mu} = 0. \tag{25}$$

Substituting, in (25), T^{ν}_{μ} and T from (24) and $T^{\nu}_{\mu;\nu}$ from (21), one obtains

$$\begin{aligned} &2(n+4)g^{\lambda\nu}\phi_{;\lambda\mu}\phi_{,\nu} - 2(n+1)g^{\lambda\nu}\phi_{;\lambda\nu}\phi_{,\mu} \\ &\quad - 3(n+2)g^{\lambda\nu}\phi_{,\lambda}\phi_{,\nu}\phi_{,\mu} - \frac{6}{a^2}[1 - e^{(n+2)\phi}]g^{\underline{\mu\nu}}\phi_{,\nu} = 0. \end{aligned} \tag{26}$$

By (4) and (5), the above equation reads in terms of γ-derivatives

$$\Big[2(n+4)g^{\lambda\nu}\phi_{|\mu\nu} - 2(n+1)g^{\nu\rho}\phi_{|\nu\rho}\delta^{\lambda}_{\mu} - (n-2)g^{\lambda\nu}\phi_{,\nu}\phi_{,\mu} - (n+4)g^{\lambda\nu}g^{\rho\sigma}g_{\nu\rho|\mu}\phi_{,\sigma} - \frac{6}{a^2}(1-e^{(n+2)\phi})g^{\lambda\underline{\mu}}\Big]\phi_{,\lambda} = 0. \tag{27}$$

Under the assumption (23), the scalar ϕ satisfies the system (26), resp. (27). A necessary condition of existence of a nonconstant ϕ implies the vanishing of the determinant of the coefficients of $\phi_{,\lambda}$ in (27).

Equation (27) is the necessary and sufficient condition, under the assumption (23), for the conservation of $\tilde{T}^{\nu}_{\mu}$ in the metric $\tilde{g}_{\mu\nu}$. To verify, we first put back (27), with the help of (4) and (5), in the form (26). Then, assuming $e^{(n+2)\phi} \neq 1$, we take the expressions of T^{ν}_{μ} and T from (24) and, multiplying them by $\phi_{,\nu}$ and the corresponding numerical coefficients, we obtain an expression for the terms of (26) involving second order derivatives of ϕ. Then, substituting $T^{\nu}_{\mu;\nu}$ from (21) in the term linear in $\phi_{,\mu}$ (the last one) in (26), we reconstruct (25). Substituting then $\tilde{T}^{\nu}_{\mu}$ from (23), we finally obtain the second equality of (19). Hence $\tilde{T}^{\nu}_{\mu}$ is conservative by (27).

Contracting (27) with $\phi_{,\mu}$, solving with respect to $\exp[(n+2)\phi]$, then substituting again in (27), one finally obtains

$$\Big\{(g^{\rho\sigma}\phi_{,\rho}\phi_{,\sigma})\,[2(n+4)g^{\lambda\nu}\phi_{|\nu\mu} - 2(n+1)g^{\nu\tau}\phi_{|\nu\tau}\delta^{\lambda}_{\mu} - (n-2)g^{\lambda\nu}\phi_{,\nu}\phi_{,\mu} - (n+4)g^{\lambda\nu}g^{\tau\chi}g_{\nu\tau|\mu}\phi_{,\chi}] - [2(n+4)g^{\rho\nu}\phi_{,\rho}\phi_{,\sigma}\phi_{|\nu\underline{\sigma}} - 2(n+1)g^{\nu\rho}\phi_{|\nu\rho}\phi_{,\sigma}\phi_{,\underline{\sigma}} - (n-2)g^{\nu\rho}\phi_{,\nu}\phi_{,\rho}\phi_{,\sigma}\phi_{|\nu\underline{\sigma}} - (n+4)g^{\nu\tau}g^{\rho\chi}g_{\nu\rho|\underline{\sigma}}\phi_{,\tau}\phi_{,\chi}\phi_{,\sigma}]g^{\lambda\underline{\mu}}\Big\}\phi_{,\lambda} = 0. \tag{28}$$

Deriving (28) we assumed $g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} \neq 0$, i.e. $\phi_{,\mu}$ could not have been a light-like vector. Substituting $\phi_{,\mu}$ from (16), (17) in (28), one obtains the condition satisfied by the metric $g_{\mu\nu}$ in the case of energy conservation in the metric $\tilde{g}_{\mu\nu}$, under the assumption (23). Further, $\phi_{,\mu}$ being a gradient, (18) must be identically satisfied.

Let us return to the expressions involving the energy tensor. By (21) and (23) the relation (22) takes the form

$$\left[\delta^\lambda_\mu - \frac{4}{3}(n+4)\pi a^2 T_{\underline{\lambda}\mu} + \frac{4}{3}\pi a^2 g_{\underline{\lambda}\mu} T\right] T^\nu_{\lambda;\nu} = 0. \tag{29}$$

The vanishing of the determinant of the expression inside the brackets is, equivalently with (27), a necessary condition of existence of a nonconstant ϕ, since T^ν_μ cannot be, by (21), conservative in that cse.

(b) Consider the case of the conformal invariance of $T_{\mu\nu}$, which holds for $n = -2$ in (23). One has from (26) [which is more convenient in this case than (27)]:

$$2g^{\lambda\nu}\phi_{;\lambda\mu}\phi_{,\nu} + g^{\lambda\nu}\phi_{;\lambda\nu}\phi_{,\mu} = 0. \tag{30}$$

Then, puting $n = -2$ in (24) and contracting with $\phi_{,\nu}$ one has

$$g^{\lambda\nu}\phi_{;\lambda\mu}\phi_{,\nu} - g^{\lambda\nu}\phi_{,\lambda}\phi_{,\mu}\phi_{,\nu} - g^{\lambda\nu}(\phi_{;\lambda\nu} + \tfrac{1}{2}\phi_{,\lambda}\phi_{,\nu})\phi_{,\mu} = 0. \tag{31}$$

There results from (30) and (31)

$$g^{\lambda\nu}(\phi_{;\lambda\nu} + \phi_{,\lambda}\phi_{,\nu})\phi_{,\mu} = 0. \tag{32}$$

One possibility is the trivial $\phi = \text{const}$. The other one results directly from (24) for $n = -2$. Substituting that solution back in (24) one obtains

$$2\phi_{;\mu\nu} - 2\phi_{,\mu}\phi_{,\nu} + g_{\mu\nu}g^{\rho\sigma}\phi_{,\rho}\phi_{,\sigma} = 0. \tag{33}$$

Contracting with $\phi_{,\lambda}$ the above equation, one obtains from it, with the help of (32), equation (30). So, in the case of the conformal invariance of $T_{\mu\nu}$, (30) results directly from (24), i.e. from the term by which the respective gravitational equations for the two metrics differ. The condition of conservation is a consequence of (24).

(c) A characteristic case is that of $n = -4$ in (23). We shall apply it to an electromagnetic field first. The energy tensor of a pure electromagnetic field, given by the skew-symmetric tensor $F_{\alpha\beta}$, reads

$$T^\nu_\mu = \frac{1}{4\pi}\left(F_{\mu\beta}F^{\nu\beta} - \frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}\delta^\nu_\mu\right). \tag{34}$$

Introducing the conformally related field by

$$\tilde{F}_{\alpha\beta} = F_{\alpha\beta}\,; \qquad \tilde{F}^{\alpha\beta} = e^{-4\phi}F^{\alpha\beta} \tag{35}$$

the energy tensor transforms as follows

$$\tilde{T}^{\nu}_{\mu} = e^{-4\phi} T^{\nu}_{\mu}. \tag{36}$$

The transformation (35) is well-known; it satisfies the requirement that Maxwell's equations remain valid after it. As the energy tensor (34) is trace-free, we have, by (25), $T^{\nu}_{\mu;\nu} = 0$. The electromagnetic energy is conservative both before and after the transformation, the result being, by (21), $\phi = \text{const}$. Another possibility would be to consider Maxwell's equations satisfied only after the conformal transformation, which is quite unrealistic. So, a pure electromagnetic field allows only trivially related conformally equivalent metrics.

Let us assume, without specifying it physically, that the energy tensor has a nonvanishing trace $T \neq 0$. This could be the case of a charged perfect fluid, the conformal transformation of T^{ν}_{μ} being given by (36). Assume the physical metric $g_{\mu\nu}$ orthogonal (the metric $\gamma_{\mu\nu}$ is taken to be orthogonal; Ref. 2). As the nongravitational energy tensor is conservative only after the transformation, the vanishing of the determinant of (29) yields four possibilities:

$$T_{(A)} = -\frac{3}{4\pi a^2} g^{AA} \qquad (A = 1, 2, 3, 4). \tag{37}$$

In the general case there is only one index A for which the above relation can be satisfied (otherwise the system $\gamma_{\mu\nu}$ would determine all the remaining metric coefficients $g_{\mu\nu}$). Therefore only one $T^{\nu}_{\mu;\nu}$, corresponding to the index of (37), can be different from zero. By (21) the respective $\phi_{(A)}$ depends on one variable only. For $A = 4$, one has a conformal factor depending only on time, allowing the expansion or the contraction of the physical metric.

As can be seen from this section, physical metrics, conformally equivalent to a given basic metric and allowing the conservation of nongravitational energy, are subject to limitations stronger than was the case in classicalgeneral relativity. An immediate conclusion, resulting from (16), is that every conservative conformally flat metric reduces, through a constant conformal factor, to the background metric $\gamma_{\mu\nu}$.

Another characteristic feature of the behavior of energy under conformal transformations results from (21). In the case of ϕ depending on one variable only, say on time, only one of the four equations (21) has a right hand side different from zero. Moreover, in the case of the linear dependence $\phi = at + b$, that term reduces to a constant. The same is true of the left hand side of (16). If we restrict ourselves further to an orthogonal physical metric, three of the dynamical equations resulting from the condition of energy conservation in the initial metric remain unchanged, the fourth one differing only by the ratio of γ_{44} to g_{44}. Equation (22) also takes a peculiar form in the case considered. Three components of the divergence of $\tilde{T}^{\nu}_{\mu}$ are null, and the fourth one becomes a linear function, with constant coefficients, of $\tilde{T}^{\nu}_{\mu}$ and its trace.

In the case of homogeneously transforming tensors, defined in (a), and physically realistic at least for simple cases of nongravitational energy, the value $n = -4$ leads, for trace-free tensors, $T = 0$, to $\phi = \text{const.}$ by (19) and (21). The other possibility is the non-conservation of energy.

4. APPLICATION TO A WEAK FIELD

In classical relativity a weak gravitational field differs from the Minkowski metric only by sufficiently small additional terms $\epsilon_{\alpha\beta}$. In bimetric general relativity it is natural to consider as weak a gravitational field which differs from the curved background metric by sufficiently small terms, in the sense of a given definition. So, we put

$$g_{\alpha\beta} = \gamma_{\alpha\beta} + \epsilon_{\alpha\beta} + \mathcal{O}(\epsilon) \Rightarrow g^{\alpha\beta} = \gamma^{\alpha\beta} - \epsilon^{\alpha\beta} \qquad (\mathcal{O}(\epsilon) = 0). \tag{38}$$

The assumptions concerning the differentiability of the metric coefficients, made at the beginning of Section 1, apply to the $\epsilon_{\alpha\beta}$'s. We assume their first derivatives continuous, the second ones admitting jumps. We demand that their orders of magnitude satisfy the requirement

$$\epsilon_{\alpha\beta} \sim \epsilon_{\alpha\beta|\gamma} \sim \epsilon_{\alpha\beta|\gamma\delta}. \tag{39}$$

Next is the assumption that $\gamma_{\lambda\mu}$, and correspondingly $\gamma^{\lambda\mu}$, do not exceed, in the domain considered, some given order of magnitude, so as not to influence the order of the $\epsilon_{\alpha\beta}$'s when raising or lowering their indices. That question is related to the choice of the coordinate system. Taking the background metric in its static form [Ref. 2, eq. (82)]

$$\epsilon\, d\sigma^2 = \left(1 - \frac{r^2}{a^2}\right) dt^2 - \frac{dr^2}{1 - (r^2/a^2)} - r^2(d\theta^2 + \sin\theta^2 d\phi^2) \qquad (r < a) \tag{40}$$

where a is the radius of the empty universe, we restrict ourselves to the domain where r_{max} is sufficiently smaller than a and r_{min} sufficiently greater than zero, so as not to have a determining influence on the order of the terms discarded. The minimal value $|\theta|_{min}$ has to be chosen accordingly. When studying a weak field, coordinate transformations would allow us to include the critical domains implied by the metric form (40). This makes a notable difference from classical relativity, where the basic coordinate system was Lorentzian, i.e. without influence on the metric operations involving a weak field.

By the preceding definitions, we have from (16) and (17)

$$\phi_{,\beta} = \tfrac{1}{2}(\epsilon_{\beta\alpha|\underline{\alpha}} - \tfrac{1}{2}\epsilon_{\alpha\underline{\alpha}|\beta}) \tag{41}$$

and by (4), (5), for the second derivatives

$$\phi_{;\beta\gamma} = \phi_{|\beta\gamma} - \tfrac{1}{2}(\epsilon_{\gamma\lambda|\beta} + \epsilon_{\lambda\beta|\gamma} - \epsilon_{\beta\gamma|\lambda})\phi_{,\underline{\lambda}}$$

which reduces, by (38) and (41), to

$$\phi_{;\beta\lambda} = \phi_{|\beta\gamma} = \tfrac{1}{2}(\epsilon_{\beta\alpha|\underline{\alpha}\gamma} - \tfrac{1}{2}\epsilon_{\alpha\underline{\alpha}|\beta\gamma}). \tag{42}$$

The definition of $\phi_{,\beta}$ implies

$$\epsilon_{\beta\alpha|\underline{\alpha}\gamma} = \epsilon_{\gamma\alpha|\underline{\alpha}\beta}. \tag{43}$$

We shall look for the coordinate conditions under which the right hand side can be effectively equated to a gradient. We first subject the coordinates to the infinitesimal transformation (Ref. 8, p. 110)

$$x'^{\sigma} = x^{\sigma} + \mu\xi^{\sigma} \tag{44}$$

where

$$\mu = \text{const.}\,; \quad \mu\xi^{\alpha} \sim \epsilon_{\beta\gamma} \qquad (\xi^{\alpha} \in C^2,\, C^4 \text{ piecewise}) \tag{45}$$

ξ^{α} being otherwise arbitrary. By definition, therefore, we have

$$\gamma_{\alpha\beta} + \epsilon_{\alpha\beta} = (\delta^{\rho}_{\alpha} + \mu\xi^{\rho}_{,\alpha})(\delta^{\sigma}_{\beta} + \mu\xi^{\sigma}_{,\beta})(\gamma'_{\rho\sigma} + \epsilon_{\rho\sigma}). \tag{46}$$

On the other hand, the infinitesimal transformation of $\gamma_{\rho\sigma}$ along the vector field ξ^{α} reads

$$\gamma'_{\rho\sigma} = \gamma_{\rho\sigma} + \mu\gamma_{\rho\sigma,\tau}\xi^{\tau}. \tag{47}$$

After the substitution of (47) one obtains for (46), by a well-known procedure,

$$\epsilon_{\alpha\beta} = \epsilon'_{\alpha\beta} + \mu(\xi^{\underline{\alpha}}_{|\beta} + \xi^{\underline{\beta}}_{|\alpha}) \tag{48}$$

and therefore

$$\epsilon_{\beta\alpha|\underline{\alpha}} = \epsilon'_{\beta\alpha|\underline{\alpha}} + \mu(\xi^{\alpha}_{|\beta\alpha} + \xi^{\beta}_{|\alpha\underline{\alpha}}). \tag{49}$$

The covariant differentiation being carried out with respect to the metric $\gamma_{\alpha\beta}$, we have the substitution

$$\xi^{\alpha}_{|\beta\alpha} = \xi^{\alpha}_{|\alpha\beta} + P_{\alpha\beta}\xi^{\alpha} \tag{50}$$

$P_{\alpha\beta}$ being obtained by contraction from (27). Then, demanding that ξ^{α} satisfy the equation

$$\mu\xi^{\beta}_{|\alpha\underline{\alpha}} = -\frac{3\mu}{a^2}\xi^{\beta} + \epsilon_{\beta\alpha|\underline{\alpha}}, \tag{51}$$

one finally obtains from (49)

$$\epsilon'_{\beta\alpha|\underline{\alpha}} = -\mu\xi^{\alpha}_{|\alpha\beta} = \chi_{,\beta}. \tag{52}$$

So, under the condition (51) on ξ^{α}, the quantity $\epsilon'_{\beta\alpha|\underline{\alpha}}$ represents the gradient of a scalar χ. In a suitable class of coordinate systems, obtained with the help of the vector field $\mu\xi^{\beta}$, satisfying (51), the relation (43) is identically satisfied.

(a) Assume that T^{ν}_{μ} satisfies (23). Considering that (52) holds and dropping the primes, one can write, by (3), (13), (41) and (42)

$$\epsilon_{\mu\alpha|\underline{\alpha}\nu} - \frac{1}{2}\epsilon_{\alpha\underline{\alpha}|\mu\nu} - \gamma_{\mu\nu}(\epsilon_{\alpha\beta|\underline{\beta\alpha}} - \frac{1}{2}\epsilon_{\alpha\underline{\alpha}|\beta\underline{\beta}}) = 8\pi T_{\mu\nu}[1 - e^{(n+2)\phi}]. \tag{53}$$

If we linearize, with the help of (38), the left hand side of (9), then substitute in (53) that expression to $T_{\mu\nu}$, we shall have

$$\begin{aligned}2\epsilon_{\mu\alpha|\underline{\alpha}\nu} - \epsilon_{\alpha\underline{\alpha}|\mu\nu} - \gamma_{\mu\nu}(2\epsilon_{\alpha\beta|\underline{\beta\alpha}} - \epsilon_{\alpha\underline{\alpha}|\beta\underline{\beta}}) \\ = -[e^{(n+2)\phi} - 1]\{\epsilon_{\alpha\underline{\alpha}|\mu\nu} + \epsilon_{\mu\nu|\alpha\underline{\alpha}} - \epsilon_{\alpha\mu|\nu\underline{\alpha}} - \epsilon_{\alpha\nu|\mu\underline{\alpha}} \\ - \gamma_{\mu\nu}(\epsilon_{\alpha\underline{\alpha}|\beta\underline{\beta}} - \epsilon_{\alpha\beta|\underline{\beta\alpha}})\}.\end{aligned} \tag{54}$$

One obtains from the above relation

$$2[e^{(n+2)\phi} - 1](\epsilon_{\alpha\underline{\alpha}|\beta\underline{\beta}} - \epsilon_{\alpha\beta|\underline{\beta\alpha}}) = -3(2\epsilon_{\alpha\beta|\underline{\beta\alpha}} - \epsilon_{\alpha\underline{\alpha}|\beta\underline{\beta}}). \tag{55}$$

If assuming $\epsilon_{\alpha\underline{\alpha}|\beta\underline{\beta}} \neq \epsilon_{\alpha\beta|\underline{\beta\alpha}}$ and $e^{(n+2)\phi} - 1$ of finite order of magnitude with respect to $\epsilon_{\alpha\beta}$, one can write by the above formula

$$2\epsilon_{\mu\alpha|\underline{\alpha}\nu} - \epsilon_{\alpha\underline{\alpha}|\mu\nu} - \gamma_{\mu\nu}(2\epsilon_{\alpha\beta|\underline{\beta\alpha}} - \epsilon_{\alpha\underline{\alpha}|\beta\underline{\beta}})$$
$$= \frac{3}{2}\,\frac{2\epsilon_{\alpha\beta|\underline{\beta\alpha}} - \epsilon_{\alpha\underline{\alpha}|\beta\underline{\beta}}}{\epsilon_{\alpha\underline{\alpha}|\beta\underline{\beta}} - \epsilon_{\alpha\beta|\underline{\beta\alpha}}}\,[\epsilon_{\alpha\underline{\alpha}|\mu\nu} + \epsilon_{\mu\nu|\alpha\underline{\alpha}} - \epsilon_{\alpha\mu|\nu\underline{\alpha}}$$
$$- \epsilon_{\alpha\nu|\mu\underline{\alpha}} - \gamma_{\mu\nu}(\epsilon_{\alpha\underline{\alpha}|\beta\underline{\beta}} - \epsilon_{\alpha\beta|\underline{\beta\alpha}})]. \tag{56}$$

In the case of the conformal invariance of $T_{\mu\nu}$ ($n = -2$) one immediately obtains from (53), resp. (54)

$$\epsilon_{\alpha\underline{\alpha}|\beta\underline{\beta}} = 2\epsilon_{\alpha\beta|\underline{\beta\alpha}}.$$

Hence

$$\epsilon_{\alpha\underline{\alpha}|\mu\nu} = 2\epsilon_{\mu\alpha|\underline{\alpha}\nu} \Leftrightarrow \phi_{|\mu\nu} = 0. \tag{57}$$

The latter of the above equations results from (42).

When considering other cases, one first has to substitute $\phi_{,\lambda}$ from (41) in (27). By the assumptions made at the beginning of the section it appears that the left hand side of (27) is ~ 0, except for the last term. $\phi =$ const. would result. But the coefficient $6/a^2$ is very small (a is the radius of the universe) and $\exp[(n+2)\phi] - 1$ can be also assumed to be small, so that it is reasonable to discard the last term in (27). Then, by the assumption made $a^{-2}(n+2)\phi \sim |\epsilon_{\alpha\beta}|$ (to the linear approximation). Thus, ϕ is not necessarily constant and has to be determined so as to satisfy (54).

(c) Another possible approach is to weaken the assumptions (39), so that

$$\epsilon_{\alpha\beta} \sim \epsilon_{\alpha\beta|\gamma} \not\sim \epsilon_{\alpha\beta|\gamma\delta}. \tag{58}$$

We assume the order of magnitude of the terms quadratic in $\epsilon_{\alpha\beta|\gamma\delta}$ (with possible discontinuities, $\epsilon_{\alpha\beta}$ being of class C^1, C^3 piecewise) non negligible. We also consider the products of the second derivatives of $\epsilon_{\alpha\beta}$ by $\phi_{,\gamma}$ as non negligible. Then (27) reduces, after substitution from (41), (42), to

$$[(n+4)\,\epsilon_{\underline{\lambda}\alpha|\underline{\alpha}\mu} - \frac{1}{2}\epsilon_{\alpha\underline{\alpha}|\underline{\lambda}\mu}) - (n+1)\,(\epsilon_{\alpha\beta|\underline{\beta\alpha}}$$
$$- \frac{1}{2}\epsilon_{\alpha\underline{\alpha}|\beta\underline{\beta}})\delta^{\lambda}_{\mu}]\,(\epsilon_{\lambda\alpha|\underline{\alpha}} - \frac{1}{2}\epsilon_{\alpha\underline{\alpha}|\lambda}) = 0. \tag{59}$$

Following the considerations of (a), the last term in (27) has been neglected. This is a consequence, first of the fact that a is a constant and keeps its order of magnitude, and secondly of the assumption that $\exp[(n+2)\phi]-1$, although not necessarily "weak", is sufficiently small.

The assumption (58) is justified only on small "islands" of energy. For the value $n=-5/2$, (59) reduces to the equation $(\phi_{,\underline{\lambda}}\phi_{,\mu})_{|\lambda}=0$.

Bimetric general relativity is, like every bimetric theory, particularly suited to weak gravitational fields. That is the consequence of the fact that the choice of the background coordinate system determines the character of the solutions of the gravitational field equations, as was remarked in Ref. 9, § 1.9. As we have seen, the definition of weak gravitational fields required the restriction to the domains of the background metric where the metrical coefficients could exert no influence on the order of magnitude of the departures $\epsilon_{\alpha\beta}$. To obtain a simplified situation, we chose the static coordinate system (40). This system is, of course, not the only suitable one. *In the domain in which the order of $\epsilon_{\alpha\beta}$ is preserved through the operations* involving the metrical coefficients, further transformations are allowed. With that limitation it is possible to determine the transformations (44), which enable us to establish the classes of coordinate systems in which $\epsilon'_{\alpha\beta}$ has the property (52), i.e. in which the right hand side of (41) represents a gradient.

Finally, the discussion following (57) led us to decide on the relative orders of magnitude of a^{-2} (the radius of the universe a is roughly 10^{28}cm; Ref. 2), of ϕ and of $\epsilon_{\alpha\beta}$. It was found that (57) in the case (a) resulted from the assumption $|\epsilon_{\alpha\beta}| \sim a^{-2}[e^{(n+2)\phi}-1]$ (and even from $|\epsilon_{\alpha\beta}| \gg a^{-2}[e^{(n+2)\phi}-1]$; the respective magnitudes could differ by several decimal orders). Thus, the mutual order of magnitude of the gravitational metric departures, of the curvature of the background and of ϕ plays an essential role in the determination of possible weak fields.

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