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Alternative Representations of the Radiant Energy Transport

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The Stellar Atmosphere Physical System

Two components:

- radiation field
- matter

Representation of the physical system:

macroscopic and microscopic description
• The physical ground of **radiative transfer** is the propagation of radiation through a medium, namely a **transport process**

• Any transport process is **characterized by the flux of a proper quantity**

• We will consider the **specific** transport process where **radiant energy** is carried on through a **medium** with which it **exchanges energy**.
Ab initio

God said, Let Newton be!
And all was light

(A. Pope)

Rays of light

The least Light, or part of Light, which may be stopp'd alone without the rest of the Light, or propagated alone, or do suffer anything alone, which the rest of the Light doth not or suffers not, I call a Ray of Light. (Optiks, 1704)

Light is made of corpuscle
The dual nature of light

Huygens wave theory (17th century) confirmed experimentally by Young and Fresnel

Maxwell's electromagnetic waves revealed by Hertz's experiments

Einstein's hypothesis of the quantum of light
Contents:

1. Electrodynamic formulation;
2. Fluid dynamic - like picture;
3. Microscopic picture;
4. The RT equation as a kinetic equation for photons;
5. The macroscopic RT coefficients;
6. Transport like a fluid dynamics process;
7. Comparison between the electrodynamic and the macroscopic picture.
1. Electrodynamic formulation
A Dynamical Theory of the Electromagnetic Field (1865)

\[ \nabla \cdot D = 4\pi \rho ; \\
\nabla \cdot B = 0 ; \\
\n\nabla \times E + \frac{1}{c} \frac{\partial B}{\partial t} = 0 ; \\
\n\n\n\nabla \times H + \frac{1}{c} \frac{\partial D}{\partial t} = \frac{4\pi}{c} J . \\
\]

War es ein Gott, Der diese Zeichen schreib?
Following Maxwell:

**magnetic and electric energy density**

\[ W_{\text{mag}} = \frac{1}{8\pi} H \cdot B ; \quad W_{\text{elec}} = \frac{1}{8\pi} D \cdot E . \]

**Energy is localized in the field.**

We adopt the **Gauss conventional system of units**, where

\[ [E] = [D] = [B] = [H] = M^{1/2} L^{-1/2} T^{-1} \]

\[ \varepsilon = \mu = 1 ; \quad [\varepsilon] = [\mu] = M^0 L^0 T^0 \]
Poynting's vector: \[ S \equiv \frac{c}{4 \pi} E \times H \]

\[
[S] = \left( L T^{-1} \right) \left( M^{1/2} L^{-1/2} T^{-1} \right)^2 = \left( M L^2 T^{-2} \right) T^{-1} L^{-2}
\]
i.e. \[ \frac{\text{energy}}{\text{time} \cdot \text{surface}} = \text{power flux} \]

By a proper treatment of the last two Maxwell's equations

\[ \frac{1}{4\pi} H \cdot \dot{B} + \frac{1}{4\pi} E \cdot \dot{D} + E \cdot J + \nabla S = 0 \]

\[ \text{Poynting's theorem} \]

\[ [\text{each term}] = M L^{-1} T^{-3} = \left( M L^2 T^{-2} \right) T^{-1} L^{-3} \quad \text{i.e. power density} \]
Joule heat: \( W_J \equiv E \cdot J \)

\[ W \equiv W_{\text{elec}} + W_{\text{mag}} \]

It can be shown that \( \dot{W}_{\text{elec}} = \frac{1}{8\pi} E \cdot \dot{D} + \frac{1}{8\pi} \dot{E} \cdot D = \frac{1}{4\pi} E \dot{D} \)

The same for \( \dot{W}_{\text{mag}} \).

Hence from Poynting's theorem

\[ \dot{W} + \nabla \cdot S = -W_J. \]

energy balance of the electromagnetic field

By integration over a volume \( V \) and Gauss theorem

\[ \int_{\Sigma} S \cdot n \, d\sigma = - \int_V \left[ \frac{\partial W}{\partial t} + W_J \right] \, dV \]

conservation equation
Physical meaning of the Poynting's vector:

energy flux per unit time across unit area of the boundary surface of the volume considered

→ Transport of energy of the electromagnetic field

The Poynting's vector accounts for the intrinsic directed aspect of the propagation of the electromagnetic field.
**Transport of radiant energy by an e.m. wave**

**monochromatic polarized plane wave**

propagating along the x-axis, specified by \( \hat{x} \)

\[
E \perp H : \quad \text{only } E_y \text{ and } H_z \text{ not } 0
\]

\[
\frac{1}{c^2} \frac{\partial^2 E_y}{\partial t^2} - \frac{\partial^2 E_y}{\partial x^2} = 0 \quad \text{wave equation}
\]

solution \( E_y(x, t) = E_0 \cos(kx - \omega t) \)

by proper manipulation

\[
\frac{\partial}{\partial t} \left\{ \frac{1}{8\pi k^2} \left[ \frac{1}{c^2} \left( \frac{\partial E_y}{\partial t} \right)^2 + \left( \frac{\partial E_y}{\partial x} \right)^2 \right] \right\} - \frac{\partial}{\partial x} \left( \frac{1}{4\pi k^2} \frac{\partial E_y}{\partial t} \frac{\partial E_y}{\partial x} \right) = 0.
\]

\[
e \equiv \frac{1}{8\pi k^2} \left[ \frac{1}{c^2} \left( \frac{\partial E_y}{\partial t} \right)^2 + \left( \frac{\partial E_y}{\partial x} \right)^2 \right]
\]

\[
f \equiv - \left( \frac{1}{4\pi k^2} \frac{\partial E_y}{\partial t} \frac{\partial E_y}{\partial x} \right)
\]
From the previous definitions:

\[ \frac{\partial e}{\partial t} + \frac{\partial f}{\partial x} = 0. \]

**wave equation**  \[\Rightarrow\]  **equation of continuity**

\[ [e] = (ML^2T^{-2})L^{-3}; \quad [f] = (ML^2T^{-2})T^{-1}L^{-2} \]

**energy density**  \[\Rightarrow\]  **power flux**

\[ e(t) = \frac{E_0^2}{4\pi} \sin^2(kx - \omega t) \]

\[ W(t) = W_{elec}(t) + W_{mag}(t) = \frac{E_0^2}{4\pi} \cos^2(kx - \omega t) \]

\[ f(t) = \frac{E_0^2}{4\pi} \frac{\omega}{k} \sin^2(kx - \omega t) = \frac{c}{4\pi} E_0^2 \sin^2(kx - \omega t) \]

\[ S(t) = \frac{c}{4\pi} E^2_y(t) \hat{x} = \frac{c}{4\pi} E_0^2 \cos^2(kx - \omega t) \hat{x}. \]
2. Fluid dynamic – like picture
Picture based on **macroscopic quantities**

related to the **microscopic photon picture**

(corpuscular model of the radiation field)

Analogue with fluid dynamics:

**macroscopic flux** of particles propagating along the

**paths of geometrical optics** (eikonal equation)

that **carry on and exchange energy** with matter particles
Ray:

amount of *radiant energy* of frequency $\nu$
carried on along the *direction* $n$ with speed $c$
per unit time

across a *unit surface* perpendicular to $n$

*rays* $\iff$ *transport of energy*
Under the assumptions of

a **weak** electromagnetic field and

propagation through a **diluted** medium

the energy carried on by rays obeys the **empirical** laws of

**radiometry**

1. propagation through vacuum along straight lines with speed \( c \);
2. all rays through a given point are **independent**;
3. they are **linearly additive** both in **direction and frequency**.

(Hypotheses already formulated by Newton in his Opticks)
The above laws of photometry warrants that

the transport process is intrinsically linear

However

a single directed quantity (i.e. a vector) is not enough to specify completely the radiation field:

virtually infinite pencil of rays
From rays to specific intensity

Fundamental **physical observable** in radiative transfer:

the **energy** carried on by a ray

⇒ Scalar **macroscopic**, **local** and **directed** quantity:

**specific intensity of the radiation field**
observable:
amount of energy \( \delta E_\nu(n) \)

elements of the measure:
oriented surface \( k \, \delta S \) around \( P_1 \)
solid angle \( \delta \Omega \) around \( n \)
time interval \( \delta t \)
spectral range \( \delta \nu \)

\[
\delta E_\nu(n) \propto n \cdot k \, \delta S \, \delta \Omega \, \delta \nu \, \delta t
\]

\[
(n \cdot k)^{-1} \lim \delta S \, \delta \Omega \, \delta \nu \, \delta t \to 0 \quad \frac{\delta E_\nu(n)}{\delta S \, \delta \Omega \, \delta \nu \, \delta t} \equiv I(r,t;n,\nu)
\]
By definition the **specific intensity** \( I(\mathbf{r}, t; n, \nu) \)

is the **coefficient of proportionality**

between the

**observable** and the **elements of the measurement**

Dimension:

\[
\lbrack I \rbrack = \left( ML^2 T^{-2} \right) \cdot L^{-2} \cdot T^{-1} \cdot T
\]

i.e. **energy flux per unit time and unit frequency band**
Moments of the specific intensity

0\textsuperscript{th} order moment: \textit{average mean intensity}

\[ J(\mathbf{r},t;\nu) \equiv \frac{1}{4\pi} \oint I(\mathbf{r},t;n,\nu) \, dn \]  \hspace{1cm} \text{scalar}

1\textsuperscript{st} order moment: \textit{flux of radiation}

\[ \mathbf{F}_\nu(\mathbf{r},t) \equiv \oint I(\mathbf{r},t;n,\nu) \, n \, dn \]  \hspace{1cm} \text{vector}

2\textsuperscript{nd} order moment: \textit{radiation pressure}

\[ \mathbf{T}_\nu(\mathbf{r},t) \equiv \frac{1}{c} \oint I(\mathbf{r},t;n,\nu) \, n \, n \, dn \]  \hspace{1cm} \text{tensor}

Dyadic notation
Energy density of the radiation field

In the time interval $dt$ the volume $dV = n \cdot k \, dS \, c \, dt$ is filled in by radiant energy

Specific energy density: $U(r,t;n,\nu) \equiv \frac{d \, E_{\nu}(n)}{dV}$

directed and spectral

By definition $U(r,t;n,\nu) \, d\Omega \, d\nu = \frac{1}{c} \, I(r,t;n,\nu) \, d\Omega \, d\nu$

By integration over all the directions

$u_{\nu} \equiv u(r,t;\nu) \equiv \frac{1}{c} \oint I(r,t;n,\nu) d\Omega = \frac{4\pi}{c} \, J(r,t;\nu)$

$[u_{\nu}] = \left( M \, L^2 \, T^{-2} \right) \, L^{-3} \, T$
3. Microscopic picture
The photon distribution function

Because of the corpuscular nature of photons, let us define a distribution function such that

\[ f(r, t; n, \nu) \, d\Omega \, d\nu \]

is equal to the nr. of photons per unit volume at \( r \) and \( t \) in the band \( (\nu, \nu + d\nu) \) that propagates along \( n \) with speed \( c \) into \( d\Omega \).

\[ f \] is characterized by the pair \( (n; \nu) \)
directed and spectral

\[ [f] = L^{-3} \cdot T \]
The number of specific photons crossing the surface \( k \cdot n \, dS \) into \( d \Omega \) during \( dt \) to fill a volume \( dV = n \cdot k \, dS \, c \, dt \)

is by definition

\[
f(r, t; n, \nu) \, n \cdot k \, dS \, c \, dt \, d \Omega \, d \nu
\]

Each photon carries on its energy \( h \nu \)

**Transport process** in terms of the **photon distribution function**

**Specific energy** flowing through \( k \cdot n \, dS \)

\[
d E_\nu(n) = h \, \nu \, c \, f(r, t; n, \nu) \, n \cdot k \, dS \, d \Omega \, d \nu \, dt
\]

By direct comparison

\[
I(r, t; n, \nu) = c \, h \, \nu \, f(r, t; n, \nu)
\]
Time for a cup of tea

(May be a pint would be better)
4. The RT equation as a kinetic equation for photons
From a **formal standpoint**

a **kinetic equation** for any **transported quantity** is formally

Total rate of change = Source terms – Sink terms

(Boltzmann's equation)

Total rate of change = Eulerian derivative:

\[
\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \frac{\partial}{\partial \mathbf{r}} + \dot{\mathbf{p}} \cdot \frac{\partial}{\partial \mathbf{p}}
\]

In our case

**Sources and sinks determined by:**

atomic properties of the interaction matter - radiation

equation of state of matter (LTE) or **SE equations**
Distribution function \( F(r, p, t) \) for photons with momentum

\[
p = n \frac{h \nu}{c} ; \quad p = p(n, \nu)
\]

**Kinetic equation:**

\[
\frac{d}{dt} F(r, p, t) = \left[ \frac{\delta F}{\delta t} \right]_{\text{sources}} - \left[ \frac{\delta F}{\delta t} \right]_{\text{sinks}}
\]

It can be shown that

\[
f(r, t; n, \nu) = \frac{h^3 \nu^2}{c^3} F(r, p, t)
\]

\[
I(r, t; n, \nu) = c h \nu f(r, t; n, \nu) \quad \text{parameters} \ (n, \nu)
\]

\[
\Rightarrow F(r, p, t) = \frac{c^2}{h^4 \nu^3} I(r, t; n, \nu)
\]
\[
\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \frac{\partial}{\partial \mathbf{r}} + \dot{\mathbf{p}} \cdot \frac{\partial}{\partial \mathbf{p}} ; \quad \frac{\partial}{\partial \mathbf{r}} = \nabla
\]

For photons \( \mathbf{v} = c \mathbf{n} \) and \( \dot{\mathbf{p}} = 0 \)

\[
\Rightarrow \quad \frac{1}{c} \frac{\partial}{\partial t} I(\mathbf{r}, t; \mathbf{n}, \nu) + \mathbf{n} \cdot \nabla I(\mathbf{r}, t; \mathbf{n}, \nu) = 
\]

\[
\frac{1}{c} \left[ \frac{\delta I}{\delta t} \right]_{\text{sources}} - \frac{1}{c} \left[ \frac{\delta I}{\delta t} \right]_{\text{sinks}} = \left[ \frac{\delta I}{\delta l} \right]_{\text{sources}} - \left[ \frac{\delta I}{\delta l} \right]_{\text{sinks}}
\]

where \( \delta l = c \delta t \) is a path length along \( \mathbf{n} \)
Radiative Transfer equation:

$$\frac{1}{c} \frac{\partial}{\partial t} I(r, t; n, \nu) + n \cdot \nabla I(r, t; n, \nu) = \left[ \frac{\delta I}{\delta l} \right]_{\text{sources}} - \left[ \frac{\delta I}{\delta l} \right]_{\text{sinks}}$$

**Mathematical formulation of a directional problem**

**In terms of the macroscopic quantity specific intensity**

For any specific intensity, characterized by the pair of parameters \((n; \nu)\)

**One specific RT equation**

Each term in the RT equation has dimension

\[ (M L^2 T^2) L^{-2} L^{-1} = M L^{-1} T^{-2} \]
From a linear to non-linear problem

Mathematical complications arise when the individual specific RT equations are coupled together through the Source and Sink terms

Non-local problems

Moreover the transport process necessarily implies non-local effects brought about by matter-radiation interaction
5. Macroscopic RT coefficients
Consistently with the macroscopic picture

we consider homogeneouse volume elements that emit and absorb radiant energy isotropically

All the physical information at atomic level is incorporated into a limited number of macroscopic coefficients
Thermal emission coefficient

\[ \Delta E_{\nu}^{th} \] 

energy emitted along \( \mathbf{n} \) by \( \Delta V \)

into \( \Delta \Omega \)
during \( \Delta t \)
in \( (\nu, \nu + \Delta \nu) \)

measurable quantity

parameters of the measure

\[ \Delta E_{\nu}^{th} \propto \Delta V \Delta \Omega \Delta \nu \Delta t \]

\[ \lim_{\Delta \sigma \to 0} \frac{\delta E_{\nu}^{th}}{\delta V \delta \Omega \delta \nu \delta t} \equiv \eta_{\nu}^{th} \]
Decrease of the specific intensity
along a path $\delta l$ in the direction $\mathbf{n}$

True absorption coefficient $a_\nu(n)$:

fraction of energy removed
converted into internal energy

$$\delta I(n) \propto I(n) \delta l, \quad \frac{\delta I(n)}{I(n)} = a_\nu(n) \delta l$$

Likewise

Scattering coefficient $\sigma_\nu(n)$:

fraction of energy removed
diverted into a different direction
Extinction coefficient

Global effect of the attenuation,

i.e., removal of photons from a given beam

\[ \chi_\nu(n) \equiv a_\nu(n) + \sigma_\nu(n) \]

\[ [\chi_\nu] = [a_\nu] = [\sigma_\nu] = L^{-1} \]
Factorization of the macroscopic coefficients

coefficient $= \text{cross section} \times \text{nr. of carriers}$

\[ a(\nu) = a_P(\nu) n_{Pa} ; \quad \sigma(\nu) = \sigma_P(\nu) n_{Ps} \]

$a_P$ and $\sigma_P$ atomic data

\[
\begin{bmatrix}
  a_P \\
  \sigma_P
\end{bmatrix} = L^2
\]

$n_{Pa}$ and $n_{Ps}$ populations density

\[
\begin{bmatrix}
  n_{Pa} \\
  n_{Ps}
\end{bmatrix} = L^{-3}
\]
6. Transport like a fluid dynamics process
Analogy between fluid dynamics and radiative transfer

Fluid dynamics considers the motion of **fluid elements** along **streamlines**

Macroscopic representation of the radiation field:

The *amount of specific energy* \( \Delta E_{\nu}(n) \) carried on along \( n \) takes the place of the **fluid elements**

Correspondence between the **equations of fluid dynamics** and the **eikonal equation** of **geometrical optics**

**streamlines** <-> **rays**
generic \textbf{scalar} quantity \quad Q = q \, N_c

q \quad \text{quantity for an individual particle}

N_c \quad \text{Nr. of carriers along } n

velocity \quad v = v \, n

\textit{associated with the} \textbf{vector} \textit{quantity} \quad Q \, v

\textit{Space – time evolution of} \quad Q(r,t) :

\[ \frac{d Q(r,t)}{dt} = \frac{\partial Q(r,t)}{\partial t} + \nabla \cdot Q(r,t) \cdot c \, v = \frac{\partial Q(r,t)}{\partial t} + \nabla \cdot [Q(r,t) \, v] . \]

\textit{if } v \textit{ constant}
If $Q$ is conserved

$$\frac{dQ(r,t)}{dt} = 0$$

$$\Rightarrow \frac{\partial Q(r,t)}{\partial t} = -\nabla \cdot [Q(r,t) v] \quad \text{continuity equation}$$

$Q v (n \cdot k) \, d\sigma$ is the flux of $Q v$ through $k \, d\sigma$

divergence theorem:

$$\iiint_V \frac{\partial Q(r,t)}{\partial t} \, dV = -\iint_{\Sigma} Q(r,t) v (n \cdot k) \, d\sigma .$$
The integral over some volume V of the time derivative of the transported scalar quantity is equal to the flux of the associated vector quantity through the boundary of V.
In the case of radiative transfer

carriers: specific photons \((n, \nu)\)

with individual energy \(h \nu\)

and momentum \(\frac{h \nu}{c} n\)

travelling along \(n\) with velocity \(c n\)

\(N_c\) is given by the photon distribution function
7. *Electrodynamical vs. macroscopical picture*
Correspondence between the specific intensity and the electric field strength

Monochromatic plane wave of frequency \( \nu_0 = 1/T \) propagating along \( n_0 = n_0(\theta_0, \phi_0) \)

\[ n_0 \equiv \hat{x} ; \quad \hat{x} \perp E \perp H \]

\[ [E] = [D] = [B] = [H] \]
The solution of the wave equation

\[ \frac{1}{c^2} \frac{\partial^2 E_y}{\partial t^2} - \frac{\partial^2 E_y}{\partial x^2} = 0 \]

is \[ E_y(x,t) = E_0 \cos(kx - \omega t) \]

From the average over \( T \) of

\[ W_{\text{elec}} \equiv \frac{1}{8\pi} E \cdot D \]

and \[ W_{\text{mag}} \equiv \frac{1}{8\pi} H \cdot B \]

\[ \implies \langle W(t) \rangle_T = \frac{E_0^2}{8\pi} \]
Corresponding specific intensity:

\[ I(Ψ, φ, ν) = I_0 \delta(Ψ - Ψ_0) \delta(φ - φ_0) \delta(ν - ν_0) \]

\[ [I] = MT^{-2}; \quad [δ(ν - ν_0)] = T; \quad [I_0] = MT^{-3} \]

From the physical standpoint

\[ \langle W(t) \rangle_T = u(r, t) \]

\[ u_ν \equiv u(r, t; ν) \equiv \frac{1}{c} \oint I(r, t; n, ν) dΩ \]

\[ \Rightarrow \quad I_0 = \frac{c}{8\pi} E_0^2 \]

\[ [I_0] = [c E_0^2] = MT^{-3} \]
Electromagnetic counter part of $F_\nu(r,t)$

$$F_\nu(r,t) \equiv \oint I(r,t;n,\nu) \, n \, d\nu$$

is the **monochromatic power flux** of the radiation field

$$E_y(x,t) = E_0 \cos(kx - \omega t) ; \quad \hat{x} \perp E \perp H ; \quad |E_0| = |H_0|$$

$$\int_0^\infty d\nu \oint d\nu I(r,t;n,\nu) \, n = I_0 \, n_0 = \frac{c}{8\pi} \, E_0^2 \, n_0$$

**bolometric vector flux**

$$\langle S(t) \rangle_T = \frac{c}{8\pi} \, E_0^2 \, n_0$$
Correspondence of the **radiative pressure** with the **Maxwell stress tensor**

\[
p(n, \nu) = \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix} = \frac{h\nu}{c} \begin{pmatrix} n_x \\ n_y \\ n_z \end{pmatrix}
\]

**moment carried on by a photon** \((n, \nu)\)

\[
T_\nu(r, t) \equiv \frac{1}{c} \oint I(r, t; n, \nu) \ n \ n \ d\ n
\]

**radiative pressure tensor**

\[
[T_\nu] = \left( M L T^{-1} \right) L^{-2} \text{ flux of momentum}
\]

\[
k \cdot T_\nu = \begin{vmatrix} \frac{1}{c} \oint I(r, t; n, \nu) \ n_x (k \cdot n) \ d\ n \\ \frac{1}{c} \oint I(r, t; n, \nu) \ n_y (k \cdot n) \ d\ n \\ \frac{1}{c} \oint I(r, t; n, \nu) \ n_z (k \cdot n) \ d\ n \end{vmatrix}
\]

**net flux** of \(p_j\) across unit area \(k\) :

\[
\oint \frac{1}{c} I(r, t; n, \nu) \ n_j (k \cdot n) \ d\ n = (k \cdot T_\nu)_j
\]
\[
(n \cdot T_{\nu})_j = \frac{1}{c} \oint I(r, t; n, \nu) n_j \, dn = \frac{1}{c} (F_{\nu})_j
\]

**net transport of** \( p \)

\[
n \cdot \frac{1}{c} F_{\nu} = n \cdot \oint \frac{h\nu}{c} f(r, t; n, \nu) c n \, dn
\]

Let us define

\[
G_{\nu}(r, t) \equiv \frac{1}{c^2} F_{\nu}(r, t)
\]

**monochromatic momentum density** of the radiation field

\[
[G_{\nu}] = (MT^{-2}) L^{-2} T^2 = (MLT^{-1}) L^{-3} T
\]
\[ G \equiv \int_0^\infty G_\nu d \nu = \frac{1}{c^2} \int_0^\infty F_\nu d \nu = \frac{1}{c^2} S \]

To cut a long story short:

\[ \frac{\partial (G_\nu)_j}{\partial t} = \frac{1}{c^2} \frac{\partial (F_\nu)_j}{\partial t} = - \nabla \cdot [(G_\nu)_j c n] \]

continuity equation

\[ \frac{\partial}{\partial t} \int_0^\infty G_\nu d \nu = - \nabla \cdot \int_0^\infty T_\nu d \nu \implies \frac{\partial G}{\partial t} = - \nabla \cdot T \]

\( (M L T^{-1}) L^{-3} T^{-1} \)

momentum density associated with the electromagnetic field:

\[ \frac{\partial G_{em}}{\partial t} = \nabla \cdot T^M \]

\( (M L T^{-1}) L^{-3} T^{-1} \)

\[ G \iff G_{em} \]
Finis coronat operam