Serb. Astron. J. № 174 (2007), 53 - 60
DOI: 10.2298/SAJ0774053P

UDC 521.19 : 523.44
Original scientific paper

THIRD DERIVATIVES OF THE INTEGRABLE PART OF AN ASTEROID HAMILTONIAN

R. Pavlović

Astronomical Observatory, Volgina 7, 11160 Belgrade 74, Serbia

(Received: April 12, 2007; Accepted: April 27, 2007)

SUMMARY: To apply the theorem of Nekhoroshev (1977) to asteroids, one first has to check whether a necessary geometrical condition is fulfilled: either convexity, or quasi-convexity, or only a 3-jet non-degeneracy. This requires computation of the derivatives of the integrable part of the corresponding Hamiltonian up to the third order over actions and a thorough analysis of their properties. In this paper we describe in detail the procedure of derivation and we give explicit expressions for the obtained derivatives.

Key words. Celestial mechanics – Minor planets, asteroids

1. INTRODUCTION

To provide an answer to the question on the stability of motion over very long time spans, one can make use of the well-known theorem of Nekhoroshev (1977). However, to apply this theorem to the real dynamical systems, e.g. in the case of asteroids (Guzzo et al. 2002), we have to first check whether the integrable part of the corresponding Hamiltonian fulfils necessary geometrical condition required by the theorem (convexity, quasi-convexity, 3-jet non-degeneracy). To do that, we need to find the derivatives up to the third order of the integrable part of the Hamiltonian over actions. The aim of this paper is to give in full the expressions for these derivatives, and to describe in detail the procedure of their computation, since they can be determined only numerically. By using these derivatives in the analysis of the structure of the phase space, we establish which geometrical conditions are fulfilled in which part of the phase space, thus making possible the application of the theorem of Nekhoroshev in the regions where (some of) these conditions are fulfilled (Pavlović and Guzzo 2007).

2. THE ASTEROID HAMILTONIAN

The Hamiltonian of an asteroid can be represented as

$$\mathcal{H} = -\frac{1}{2L^2} + \mathcal{G}\sum_{j=1}^{N} m_j \left(\frac{1}{\Delta_j} - \frac{\mathbf{r}\cdot\mathbf{s}_j}{\mathbf{s}_j^3}\right), \qquad (1)$$

where Δ_j is the distance of the j-th planet from the asteroid, $\mathbf{s}_j$ is the distance of the j-th planet from the Sun, m_j is the mass of the j-th planet, $j = 1, \ldots, 8$, and $\mathcal{G}$ is the gravitational constant.

Introducing the modified Delaunay's variables

$$\begin{aligned} \Lambda &= L, & \lambda &= l + \varpi, \\ H &= L - G, & h &= -\varpi, \\ Z &= G - \Theta, & \zeta &= -\vartheta, \end{aligned} \qquad (2)$$

the Hamiltonian (1) becomes

$$\mathcal{H} = -\frac{1}{2\Lambda^2} + \varepsilon\mathcal{K}^p(\Lambda, H, Z, \lambda, h, \zeta, a_j, \lambda_j, \xi_j, \eta_j, p_j, q_j), \tag{3}$$

where we select units for mass, length and time in such a way to get $\mathcal{G} = m_0 = 1$ (m_0 is mass of the Sun); λ_j are mean longitudes of the planets, ξ_j, η_j, p_j, q_j corresponding elliptic elements of the planets (Morbidelli 2002) for which the linear theory gives

$$\begin{aligned}
\xi_j &= e_j \cos\varpi_j = \sum_{k=1}^{8} M_{j,k}\cos(g_k t + \alpha_k),\\
\eta_j &= -e_j \sin\varpi_j = -\sum_{k=1}^{8} M_{j,k}\sin(g_k t + \alpha_k),\\
p_j &= \sin\frac{i_j}{2}\cos\vartheta_j = \sum_{k=1}^{8} N_{j,k}\cos(s_k t + \beta_k),\\
q_j &= -\sin\frac{i_j}{2}\sin\vartheta_j = -\sum_{k=1}^{8} N_{j,k}\sin(s_k t + \beta_k),
\end{aligned} \tag{4}$$

where g_k, s_k are the frequencies, $M_{j,k}$, $N_{j,k}$ amplitudes and α_k, β_k phases of forced perturbing terms.

Hamiltonian (3) is given as a sum of the integrable part $h_0 = \frac{1}{2\Lambda^2}$, which corresponds to the two-body problem, and of the perturbation $\varepsilon\mathcal{K}^p$, which is of the order of the mass ratio of Jupiter and the Sun ($\approx 10^{-3}$). Perturbation is time dependent indirectly, via $\lambda_j, \xi_j, \eta_j, p_j, q_j$, which are known functions of time, and of proper frequencies.

In general, when the Hamiltonian is time dependent, the space of phases can be extended by a new action conjugate to time, to get an autonomous system. Applying this technique, Hamiltonian (3) becomes

$$\begin{aligned}
\mathcal{H} = & -\frac{1}{2\Lambda^2} + n_j\Lambda_j + \varepsilon\nu_{g_j}\Lambda_{g_j} + \varepsilon\nu_{s_j}\Lambda_{s_j} +\\
& +\varepsilon\mathcal{K}^p(\Lambda, H, Z, \lambda, h, \zeta, \lambda_j,\\
& \xi_j(\lambda_{g_j}), \eta_j(\lambda_{g_j}), p_j(\lambda_{s_j}), q_j(\lambda_{s_j})),
\end{aligned} \tag{5}$$

where the frequencies of the system are explicitly present. In (5), Λ_j are conjugate to λ_j (the frequency of which are n_j), and $\Lambda_{g_j}, \Lambda_{s_j}$ are conjugate to $\lambda_{g_j} = -g_j t - \alpha_j$, that is to $\lambda_{s_j} = -s_j t - \beta_j$; we put $\varepsilon\nu_{g_j} = -g_j$ and $\varepsilon\nu_{s_j} = -s_j$ to indicate the order of magnitude of the secular frequencies.

Moreover, the Hamiltonian (5) must be expanded in Taylor series around zero in terms of elliptic elements ξ_j, η_j, p_j, q_j, because these are small quantities according to the theory of planetary motion. In this way, we get Hamiltonian in the form

$$\begin{aligned}
\mathcal{H} = & -\frac{1}{2\Lambda^2} + n_j\Lambda_j + \varepsilon\nu_{g_j}\Lambda_{g_j} + \varepsilon\nu_{s_j}\Lambda_{s_j} +\\
& +\varepsilon\mathcal{K}_0 + \varepsilon^2\mathcal{K}_1 + \cdots,
\end{aligned} \tag{6}$$

where the index i in $\mathcal{K}_i$ denotes order of the polynomial in ξ_j, η_j, p_j, q_j.

Considering region of the phase space far from the mean motion resonances between the asteroid and the planets, we can eliminate from the Hamiltonian the fast variables λ i λ_j, by means of the Lie's algorithm. Hamiltonian (6) then reads

$$\begin{aligned}
\mathcal{H}_A = & -\frac{1}{2\Lambda_m^2} + n_j\Lambda_{j,m} + \varepsilon\nu_{g_j}\Lambda_{g_j,m} +\\
& +\varepsilon\nu_{s_j}\Lambda_{s_j,m} + \varepsilon\overline{\mathcal{K}_0} + \varepsilon^2\overline{\mathcal{K}_1} + \cdots,
\end{aligned} \tag{7}$$

where bar denotes averaging over λ and λ_j, which can be performed by computing the double integral. As a results we do not get an analytic expression for the Hamiltonian, but we can numerically compute it, as well as its derivatives, for every point in the phase space.

The integrable part of the Hamiltonian (7),

$$\mathcal{H}_0 = -\frac{1}{2\Lambda^2} + \varepsilon\overline{\mathcal{K}_0}, \tag{8}$$

is a function of actions Λ, H, Z and of the angle $g = \zeta - h$ (Kozai 1962), where, for simplicity, we omit the index $_m$. It is now necessary to apply Henrard's (1990) semi numeric method to the Hamiltonian (8) to reduce it to the form independent of the angle g

$$\mathcal{H}_0(\Lambda, J, Z) = h_0 + \varepsilon K_0(\Lambda, J, Z), \tag{9}$$

where Λ, J, Z are the actions. In practice, this is the easiest to do by switching from variables (2), via

$$\begin{aligned}
P &= L - \Theta, & p &= -\vartheta - g,\\
Q &= G - \Theta, & q &= g,
\end{aligned} \tag{10}$$

to cartesian canonical variables

$$\begin{aligned}
x &= \sqrt{2Q}\cos q,\\
y &= \sqrt{2Q}\sin q,
\end{aligned} \tag{11}$$

and then to action-angle variables

$$\begin{aligned}
y &= Y(\psi, J, Z),\\
x &= X(\psi, J, Z),\\
p &= z + \varrho(\psi, J, Z),\\
P &= Z,
\end{aligned} \tag{12}$$

where J and Z are the actions, ψ and z their conjugate angles, and ϱ is correction which preserves canonical character of the transformation (12).

3. DERIVATIVES OF THE INTEGRABLE PART OF THE HAMILTONIAN

Proper frequencies of the integrable part of the Hamiltonian (9) are:

$$\begin{aligned}\omega_1 &= \frac{\partial h_0}{\partial \Lambda} + \frac{\partial(\varepsilon K_0)}{\partial \Lambda},\\ \omega_2 &= \frac{\partial(\varepsilon K_0)}{\partial J} = \frac{2\pi}{T},\\ \omega_3 &= \frac{\partial(\varepsilon K_0)}{\partial Z} = \frac{1}{T}\int_0^T \frac{\partial(\varepsilon K_0)}{\partial P} dt.\end{aligned} \tag{13}$$

Frequencies ω_1, ω_2, and ω_3 can be computed numerically. T is the period which corresponds to the periodic orbit in the (x, y) plane; the integration is performed along that periodic curve (γ) (Fig. 1). Problem of finding the curve (γ) can be solved in various ways; we have chosen to integrate numerically equations of motion of the Hamiltonian K_0. Even if this method consumes more computing time, it has the advantage of being equally applicable in case of the circulation, as well as in case of the libration of the angle g.

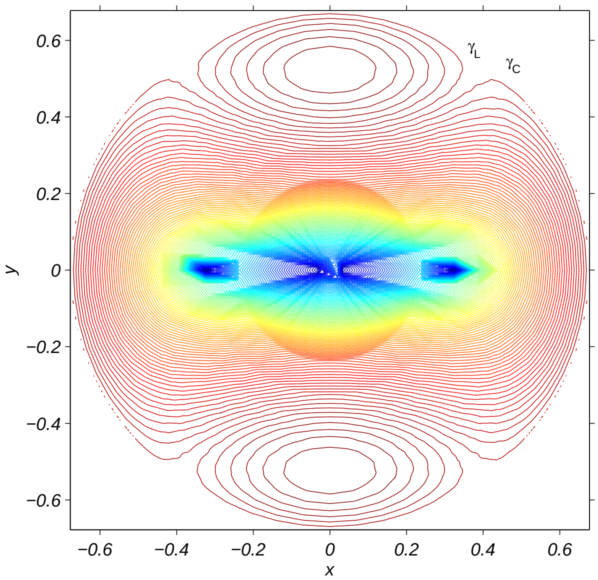

Fig. 1. *Level curves of Kozai's Hamiltonian K_0 in the (x, y) plane. Curve γ_L corresponds to the periodic orbit for libration, and γ_C for circulation of the angle g.*

Derivatives of the frequency ω_1 over actions Λ, J, and Z can also be computed numerically. For example, derivative of ω_1 over Λ can be computed directly from

$$\frac{\partial \omega_1}{\partial \Lambda} = \frac{\partial^2 \mathcal{H}_0}{\partial \Lambda^2} = -\frac{3}{\Lambda^4} + \frac{\partial^2(\varepsilon K_0)}{\partial \Lambda^2} \tag{14}$$

while the derivatives over J and Z are:

$$\frac{\partial \omega_1}{\partial J} = \frac{\partial^2 \mathcal{H}_0}{\partial \Lambda \partial J} = \frac{\partial^2(\varepsilon K_0)}{\partial \Lambda \partial J} = -\frac{2\pi}{T^2}\frac{\partial T}{\partial \Lambda}, \tag{15}$$

$$\begin{aligned}\frac{\partial \omega_1}{\partial Z} &= \frac{\partial^2 \mathcal{H}_0}{\partial \Lambda \partial Z} = \frac{\partial^2(\varepsilon K_0)}{\partial \Lambda \partial Z} =\\ &= \frac{1}{T}\int_0^T \frac{\partial^2(\varepsilon K_0)}{\partial \Lambda \partial P} dt -\\ &\quad - \left[\omega_3 - \frac{\partial(\varepsilon K_0)}{\partial P}(T)\right] \frac{1}{T}\frac{\partial T}{\partial \Lambda}.\end{aligned} \tag{16}$$

We see that in the above expressions the derivative $\frac{\partial T}{\partial \Lambda}$ appears, which represents change of the period of the periodic orbit with the change of parameter Λ, that is of the semimajor axis. By differentiating expression for the period (Morbidelli and Henrard 1991)

$$T = \int_\gamma \left(\frac{\partial(\varepsilon K_0)}{\partial Q}\right)^{-1} dg, \tag{17}$$

we straightforwardly find

$$\frac{\partial T}{\partial \Lambda} = -\int_\gamma \left(\frac{\partial(\varepsilon K_0)}{\partial Q}\right)^{-2} \left(\frac{\partial^2(\varepsilon K_0)}{\partial \Lambda \partial Q}\right) dg, \tag{18}$$

where the integration is performed along the periodic curve γ.

According to Eq. (15) for the derivative of the frequency ω_2 over Λ, one gets

$$\frac{\partial \omega_2}{\partial \Lambda} = \frac{\partial^2 \mathcal{H}_0}{\partial \Lambda \partial J} = \frac{\partial \omega_1}{\partial J} = -\frac{2\pi}{T^2}\frac{\partial T}{\partial \Lambda}. \tag{19}$$

The remaining derivatives of the frequency ω_2 over actions J and Z are explicitly given in Henrard (1990)

$$\frac{\partial \omega_2}{\partial J} = \frac{\partial}{\partial J}\left(\frac{2\pi}{T}\right) = -\frac{2\pi}{T^2}\frac{\partial T}{\partial J}, \tag{20}$$

$$\frac{\partial \omega_2}{\partial Z} = \frac{\partial}{\partial Z}\left(\frac{2\pi}{T}\right) = -\frac{2\pi}{T^2}\frac{\partial T}{\partial Z}. \tag{21}$$

Derivatives of the period can be estimated on the basis of periodicity of the solutions $x = X(t, J, Z)$ or $y = Y(t, J, Z)$, that is

$$X(0, J, Z) = X(T, J, Z), \tag{22}$$

while by differentiating (22) over J and Z one gets

$$\begin{aligned}
\dot{x}(0)\frac{\partial T}{\partial J} &= \frac{\partial X}{\partial J}(0) - \frac{\partial X}{\partial J}(T),\\
\dot{x}(0)\frac{\partial T}{\partial Z} &= \frac{\partial X}{\partial Z}(0) - \frac{\partial X}{\partial Z}(T).
\end{aligned} \tag{23}$$

If $\dot{x}(0) = 0$, then one has to use the property of the periodicity of the solution $Y(t, J, Z)$

$$\dot{y}(0)\frac{\partial T}{\partial J} = \frac{\partial Y}{\partial J}(0) - \frac{\partial Y}{\partial J}(T), \tag{24}$$

since $\dot{x}(0)$ and $\dot{y}(0)$ cannot simultaneously be equal to zero (Henrard, 1990). For the derivative of the frequency ω_3 over Λ, according to Eq. (16), one finds

$$\begin{aligned}
\frac{\partial \omega_3}{\partial \Lambda} &= \frac{\partial^2 \mathcal{H}_0}{\partial \Lambda \partial Z} = \frac{\partial \omega_1}{\partial Z} =\\
&= \frac{1}{T}\int_0^T \left[\frac{\partial^2(\varepsilon K_0)}{\partial \Lambda \partial P} + \frac{\partial^2(\varepsilon K_0)}{\partial x \partial P}\frac{\partial x}{\partial \Lambda} + \frac{\partial^2(\varepsilon K_0)}{\partial y \partial P}\frac{\partial y}{\partial \Lambda}\right] dt - \left[\omega_3 - \frac{\partial(\varepsilon K_0)}{\partial P}(T)\right]\frac{1}{T}\frac{\partial T}{\partial \Lambda}.
\end{aligned} \tag{25}$$

Computation of $\frac{\partial \omega_3}{\partial J}$ and $\frac{\partial \omega_3}{\partial Z}$ is slightly more complicated

$$\begin{aligned}
\frac{\partial \omega_3}{\partial J} &= \frac{\partial}{\partial J}\left[\frac{1}{T}\int_0^T \frac{\partial(\varepsilon K_0)}{\partial P} dt\right] =\\
&= -\left[\omega_3 - \frac{\partial(\varepsilon K_0)}{\partial P}(T)\right]\frac{1}{T}\frac{\partial T}{\partial J} + \frac{1}{T}\int_0^T \left[\frac{\partial^2(\varepsilon K_0)}{\partial x \partial P}\frac{\partial x}{\partial J} + \frac{\partial^2(\varepsilon K_0)}{\partial y \partial P}\frac{\partial y}{\partial J}\right] dt,
\end{aligned} \tag{26}$$

$$\begin{aligned}
\frac{\partial \omega_3}{\partial Z} &= \frac{\partial}{\partial Z}\left[\frac{1}{T}\int_0^T \frac{\partial(\varepsilon K_0)}{\partial P} dt\right] =\\
&= -\left[\omega_3 - \frac{\partial(\varepsilon K_0)}{\partial P}(T)\right]\frac{1}{T}\frac{\partial T}{\partial Z} + \frac{1}{T}\int_0^T \left[\frac{\partial^2(\varepsilon K_0)}{\partial x \partial P}\frac{\partial x}{\partial Z} + \frac{\partial^2(\varepsilon K_0)}{\partial y \partial P}\frac{\partial y}{\partial Z} + \frac{\partial^2(\varepsilon K_0)}{\partial P^2}\right] dt.
\end{aligned} \tag{27}$$

Quantities $\frac{\partial x}{\partial J}$ and $\frac{\partial y}{\partial J}$ are obtained by numerically solving homogeneous variational equations

$$\begin{aligned}
\frac{d}{dt}\frac{\partial x}{\partial J} &= -\frac{\partial^2(\varepsilon K_0)}{\partial x \partial y}\frac{\partial x}{\partial J} - \frac{\partial^2(\varepsilon K_0)}{\partial y^2}\frac{\partial y}{\partial J},\\
\frac{d}{dt}\frac{\partial y}{\partial J} &= \frac{\partial^2(\varepsilon K_0)}{\partial x^2}\frac{\partial x}{\partial J} + \frac{\partial^2(\varepsilon K_0)}{\partial x \partial y}\frac{\partial y}{\partial J},
\end{aligned} \tag{28}$$

where, for the initial conditions (Henrard and Lemaitre 1986), one should take $(1, \frac{\partial y}{\partial x}(0))$, while derivatives $\frac{\partial x}{\partial Z}$ and $\frac{\partial y}{\partial Z}$ are computed by solving the non homogeneous variational equations

$$\begin{aligned}
\frac{d}{dt}\frac{\partial x}{\partial Z} &= -\frac{\partial^2(\varepsilon K_0)}{\partial x \partial y}\frac{\partial x}{\partial Z} - \frac{\partial^2(\varepsilon K_0)}{\partial y^2}\frac{\partial y}{\partial Z} - \frac{\partial^2(\varepsilon K_0)}{\partial P \partial y}\\
\frac{d}{dt}\frac{\partial y}{\partial Z} &= \frac{\partial^2(\varepsilon K_0)}{\partial x^2}\frac{\partial x}{\partial Z} + \frac{\partial^2(\varepsilon K_0)}{\partial x \partial y}\frac{\partial y}{\partial Z} + \frac{\partial^2(\varepsilon K_0)}{\partial P \partial x}
\end{aligned} \tag{29}$$

with the initial condition $(0, 0)$.

In total, there are ten various third derivatives of an integrable asteroid Hamiltonian (9) over actions (Λ, J, Z), needed to check the fulfillment of the 3-jet non degeneracy condition. Directly from (14) we compute

$$\frac{\partial^2 \omega_1}{\partial \Lambda^2} = \frac{\partial^3 \mathcal{H}_0}{\partial \Lambda^3} = \frac{12}{\Lambda^5} + \frac{\partial^3(\varepsilon K_0)}{\partial \Lambda^3}, \tag{30}$$

while from (19) and (25) we get somewhat more complex expressions

$$\frac{\partial^2 \omega_2}{\partial \Lambda^2} = \frac{4\pi}{T^3}\left(\frac{\partial T}{\partial \Lambda}\right)^2 - \frac{2\pi}{T^2}\frac{\partial^2 T}{\partial \Lambda^2}, \tag{31}$$

$$\begin{aligned}\frac{\partial^2 \omega_3}{\partial \Lambda^2} &= \frac{\partial^2}{\partial \Lambda^2}\left(\frac{1}{T}\int_0^T \frac{\partial(\varepsilon K_0)}{\partial P}dt\right) = -\frac{\partial \omega_3}{\partial \Lambda}\frac{1}{T}\frac{\partial T}{\partial \Lambda} + \left[\omega_3 - \frac{\partial(\varepsilon K_0)}{\partial P}(T)\right]\left[\frac{1}{T^2}\left(\frac{\partial T}{\partial \Lambda}\right)^2 - \frac{1}{T}\frac{\partial^2 T}{\partial \Lambda^2}\right] - \\
&-\frac{1}{T^2}\frac{\partial T}{\partial \Lambda}\int_0^T\left[\frac{\partial^2(\varepsilon K_0)}{\partial \Lambda \partial P} + \frac{\partial^2(\varepsilon K_0)}{\partial x \partial P}\frac{\partial x}{\partial \Lambda} + \frac{\partial^2(\varepsilon K_0)}{\partial y \partial P}\frac{\partial y}{\partial \Lambda}\right]dt + \frac{2}{T}\frac{\partial T}{\partial \Lambda}\left[\frac{\partial^2(\varepsilon K_0)}{\partial \Lambda \partial P} + \frac{\partial^2(\varepsilon K_0)}{\partial x \partial P}\frac{\partial x}{\partial \Lambda} + \frac{\partial^2(\varepsilon K_0)}{\partial y \partial P}\frac{\partial y}{\partial \Lambda}\right](T) + \\
&+\frac{1}{T}\int_0^T\left[\frac{\partial^3(\varepsilon K_0)}{\partial \Lambda^2 \partial P} + \frac{\partial^3(\varepsilon K_0)}{\partial \Lambda \partial x \partial P}\frac{\partial x}{\partial \Lambda} + \frac{\partial^3(\varepsilon K_0)}{\partial \Lambda \partial y \partial P}\frac{\partial y}{\partial \Lambda} + \left(\frac{\partial^3(\varepsilon K_0)}{\partial \Lambda \partial x \partial P} + \frac{\partial^3(\varepsilon K_0)}{\partial x^2 \partial P}\frac{\partial x}{\partial \Lambda} + \frac{\partial^3(\varepsilon K_0)}{\partial x \partial y \partial P}\frac{\partial y}{\partial \Lambda}\right)\frac{\partial x}{\partial \Lambda} + \right. \\
&\left.+\left(\frac{\partial^3(\varepsilon K_0)}{\partial \Lambda \partial y \partial P} + \frac{\partial^3(\varepsilon K_0)}{\partial x \partial y \partial P}\frac{\partial x}{\partial \Lambda} + \frac{\partial^3(\varepsilon K_0)}{\partial y^2 \partial P}\frac{\partial y}{\partial \Lambda}\right)\frac{\partial y}{\partial \Lambda} + \frac{\partial^2(\varepsilon K_0)}{\partial x \partial P}\frac{\partial^2 x}{\partial \Lambda^2} + \frac{\partial^2(\varepsilon K_0)}{\partial y \partial P}\frac{\partial^2 y}{\partial \Lambda^2}\right]dt.\end{aligned} \tag{32}$$

By differentiating (19) over J and Z, we get mixed derivatives

$$\frac{\partial^2 \omega_2}{\partial \Lambda \partial J} = \frac{4\pi}{T^3}\frac{\partial T}{\partial \Lambda}\frac{\partial T}{\partial J} - \frac{2\pi}{T^2}\frac{\partial^2 T}{\partial \Lambda \partial J}, \tag{33}$$

$$\frac{\partial^2 \omega_2}{\partial \Lambda \partial Z} = \frac{4\pi}{T^3}\frac{\partial T}{\partial \Lambda}\frac{\partial T}{\partial Z} - \frac{2\pi}{T^2}\frac{\partial^2 T}{\partial \Lambda \partial Z}, \tag{34}$$

and differentiation of (27) over Λ gives

$$\begin{aligned}\frac{\partial^2 \omega_3}{\partial \Lambda \partial Z} &= -\left[\frac{\partial \omega_3}{\partial \Lambda} - \left(\frac{\partial^2(\varepsilon K_0)}{\partial \Lambda \partial P} + \frac{\partial^2(\varepsilon K_0)}{\partial x \partial P}\frac{\partial x}{\partial \Lambda} + \frac{\partial^2(\varepsilon K_0)}{\partial y \partial P}\frac{\partial y}{\partial \Lambda}\right)(T)\right]\frac{1}{T}\frac{\partial T}{\partial Z} + \left[\omega_3 - \frac{\partial(\varepsilon K_0)}{\partial P}(T)\right] \times \\
&\times\left[\frac{1}{T^2}\frac{\partial T}{\partial \Lambda}\frac{\partial T}{\partial Z} - \frac{1}{T}\frac{\partial^2 T}{\partial \Lambda \partial Z}\right] - \frac{1}{T^2}\frac{\partial T}{\partial \Lambda}\int_0^T\left[\frac{\partial^2(\varepsilon K_0)}{\partial x \partial P}\frac{\partial x}{\partial Z} + \frac{\partial^2(\varepsilon K_0)}{\partial y \partial P}\frac{\partial y}{\partial Z} + \frac{\partial^2(\varepsilon K_0)}{\partial P^2}\right]dt + \\
&+\left[\frac{\partial^2(\varepsilon K_0)}{\partial x \partial P}\frac{\partial x}{\partial Z} + \frac{\partial^2(\varepsilon K_0)}{\partial y \partial P}\frac{\partial y}{\partial Z} + \frac{\partial^2(\varepsilon K_0)}{\partial P^2}\right](T)\frac{1}{T}\frac{\partial T}{\partial \Lambda} + \frac{1}{T}\int_0^T\left[\frac{\partial^3(\varepsilon K_0)}{\partial \Lambda \partial P^2} + \right. \\
&+\frac{\partial^3(\varepsilon K_0)}{\partial x \partial P^2}\frac{\partial x}{\partial \Lambda} + \frac{\partial^3(\varepsilon K_0)}{\partial y \partial P^2}\frac{\partial y}{\partial \Lambda} + \left(\frac{\partial^3(\varepsilon K_0)}{\partial \Lambda \partial x \partial P} + \frac{\partial^3(\varepsilon K_0)}{\partial x^2 \partial P}\frac{\partial x}{\partial \Lambda} + \frac{\partial^3(\varepsilon K_0)}{\partial x \partial y \partial P}\frac{\partial y}{\partial \Lambda}\right)\frac{\partial x}{\partial Z} + \\
&\left.+\left(\frac{\partial^3(\varepsilon K_0)}{\partial \Lambda \partial y \partial P} + \frac{\partial^3(\varepsilon K_0)}{\partial x \partial y \partial P}\frac{\partial x}{\partial \Lambda} + \frac{\partial^3(\varepsilon K_0)}{\partial y^2 \partial P}\frac{\partial y}{\partial \Lambda}\right)\frac{\partial y}{\partial Z} + \frac{\partial^2(\varepsilon K_0)}{\partial x \partial P}\frac{\partial^2 x}{\partial \Lambda \partial Z} + \frac{\partial^2(\varepsilon K_0)}{\partial y \partial P}\frac{\partial^2 y}{\partial \Lambda \partial Z}\right]dt.\end{aligned} \tag{35}$$

From (20) we compute

$$\frac{\partial^2 \omega_2}{\partial J^2} = \frac{4\pi}{T^3}\left(\frac{\partial T}{\partial J}\right)^2 - \frac{2\pi}{T^2}\frac{\partial^2 T}{\partial J^2} \tag{36}$$

while differentiation of (26) over J and Z yields

$$\begin{aligned}\frac{\partial^2\omega_3}{\partial J^2} &= -\frac{\partial\omega_3}{\partial J}\frac{1}{T}\frac{\partial T}{\partial J}+\left[\omega_3-\frac{\partial(\varepsilon K_0)}{\partial P}(T)\right]\frac{1}{T^2}\left(\frac{\partial T}{\partial J}\right)^2-\left[\omega_3-\frac{\partial(\varepsilon K_0)}{\partial P}(T)\right]\frac{1}{T}\frac{\partial^2 T}{\partial J^2}- \\ &\quad -\frac{1}{T^2}\frac{\partial T}{\partial J}\int_0^T\left[\frac{\partial^2(\varepsilon K_0)}{\partial x\partial P}\frac{\partial x}{\partial J}+\frac{\partial^2(\varepsilon K_0)}{\partial y\partial P}\frac{\partial y}{\partial J}\right]dt+\left[\frac{\partial^2(\varepsilon K_0)}{\partial x\partial P}\frac{\partial x}{\partial J}+\frac{\partial^2(\varepsilon K_0)}{\partial y\partial P}\frac{\partial y}{\partial J}\right](T)\frac{1}{T}\frac{\partial T}{\partial J}+ \\ &\quad +\frac{1}{T}\int_0^T\left[\frac{\partial^3(\varepsilon K_0)}{\partial^2 x\partial P}\left(\frac{\partial x}{\partial J}\right)^2+2\frac{\partial^3(\varepsilon K_0)}{\partial x\partial y\partial P}\frac{\partial x}{\partial J}\frac{\partial y}{\partial J}+\frac{\partial^3(\varepsilon K_0)}{\partial y^2\partial P}\left(\frac{\partial y}{\partial J}\right)^2+\frac{\partial^2(\varepsilon K_0)}{\partial x\partial P}\frac{\partial^2 x}{\partial J^2}+\frac{\partial^2(\varepsilon K_0)}{\partial y\partial P}\frac{\partial^2 y}{\partial J^2}\right]dt,\end{aligned} \tag{37}$$

that is

$$\begin{aligned}\frac{\partial^2\omega_3}{\partial J\partial Z} &= -\frac{\partial\omega_3}{\partial Z}\frac{1}{T}\frac{\partial T}{\partial J}+\left[\omega_3-\frac{\partial(\varepsilon K_0)}{\partial P}(T)\right]\frac{1}{T^2}\frac{\partial T}{\partial J}\frac{\partial T}{\partial Z}-\left[\omega_3-\frac{\partial(\varepsilon K_0)}{\partial P}(T)\right]\frac{1}{T}\frac{\partial^2 T}{\partial J\partial Z}- \\ &\quad -\frac{1}{T^2}\frac{\partial T}{\partial Z}\int_0^T\left[\frac{\partial^2(\varepsilon K_0)}{\partial x\partial P}\frac{\partial x}{\partial J}+\frac{\partial^2(\varepsilon K_0)}{\partial y\partial P}\frac{\partial y}{\partial J}\right]dt+\left[\frac{\partial^2(\varepsilon K_0)}{\partial x\partial P}\frac{\partial x}{\partial J}+\frac{\partial^2(\varepsilon K_0)}{\partial y\partial P}\frac{\partial y}{\partial J}\right](T)\frac{1}{T}\frac{\partial T}{\partial Z}+ \\ &\quad +\frac{1}{T}\int_0^T\left[\frac{\partial^3(\varepsilon K_0)}{\partial^2 x\partial P}\frac{\partial x}{\partial J}\frac{\partial x}{\partial Z}+\frac{\partial^3(\varepsilon K_0)}{\partial x\partial y\partial P}\left(\frac{\partial x}{\partial J}\frac{\partial y}{\partial Z}+\frac{\partial x}{\partial Z}\frac{\partial y}{\partial J}\right)+\frac{\partial^3(\varepsilon K_0)}{\partial y^2\partial P}\frac{\partial y}{\partial J}\frac{\partial y}{\partial Z}+\frac{\partial^3(\varepsilon K_0)}{\partial x\partial P^2}\frac{\partial x}{\partial J}+\right. \\ &\quad \left.+\frac{\partial^3(\varepsilon K_0)}{\partial y\partial P^2}\frac{\partial y}{\partial J}+\frac{\partial^2(\varepsilon K_0)}{\partial x\partial P}\frac{\partial^2 x}{\partial J\partial Z}+\frac{\partial^2(\varepsilon K_0)}{\partial y\partial P}\frac{\partial^2 y}{\partial J\partial Z}\right]dt.\end{aligned} \tag{38}$$

The last and the most complicated expression one gets by differentiating (27) over Z

$$\begin{aligned}\frac{\partial^2\omega_3}{\partial Z^2} &= -\frac{\partial\omega_3}{\partial Z}\frac{1}{T}\frac{\partial T}{\partial Z}+\left[\omega_3-\frac{\partial(\varepsilon K_0)}{\partial P}(T)\right]\frac{1}{T^2}\left(\frac{\partial T}{\partial Z}\right)^2-\left[\omega_3-\frac{\partial(\varepsilon K_0)}{\partial P}(T)\right]\frac{1}{T}\frac{\partial^2 T}{\partial Z^2}- \\ &\quad -\frac{1}{T^2}\frac{\partial T}{\partial Z}\int_0^T\left[\frac{\partial^2(\varepsilon K_0)}{\partial x\partial P}\frac{\partial x}{\partial Z}+\frac{\partial^2(\varepsilon K_0)}{\partial y\partial P}\frac{\partial y}{\partial Z}+\frac{\partial^2(\varepsilon K_0)}{\partial P^2}\right]dt+ \\ &\quad +\frac{1}{T}\int_0^T\left[\frac{\partial^3(\varepsilon K_0)}{\partial^2 x\partial P}\left(\frac{\partial x}{\partial Z}\right)^2+2\frac{\partial^3(\varepsilon K_0)}{\partial x\partial y\partial P}\frac{\partial x}{\partial Z}\frac{\partial y}{\partial Z}+\frac{\partial^3(\varepsilon K_0)}{\partial y^2\partial P}\left(\frac{\partial y}{\partial Z}\right)^2+\frac{\partial^3(\varepsilon K_0)}{\partial x\partial P^2}\frac{\partial x}{\partial Z}+\right. \\ &\quad \left.+\frac{\partial^3(\varepsilon K_0)}{\partial y\partial P^2}\frac{\partial y}{\partial Z}+\frac{\partial^3(\varepsilon K_0)}{\partial P^3}+\frac{\partial^2(\varepsilon K_0)}{\partial x\partial P}\frac{\partial^2 x}{\partial Z^2}+\frac{\partial^2(\varepsilon K_0)}{\partial y\partial P}\frac{\partial^2 y}{\partial Z^2}\right]dt+ \\ &\quad +\left[\frac{\partial^2(\varepsilon K_0)}{\partial x\partial P}\frac{\partial x}{\partial Z}+\frac{\partial^2(\varepsilon K_0)}{\partial y\partial P}\frac{\partial y}{\partial Z}+\frac{\partial^2(\varepsilon K_0)}{\partial P^2}\right](T)\frac{1}{T}\frac{\partial T}{\partial Z}.\end{aligned} \tag{39}$$

The second derivatives of period T over actions Λ, J, and Z are obtained by differentiating (18). The second derivatives of coordinates (x,y) are found by differentiating variational Eqs. (28) over J

$$\begin{aligned}\frac{d}{dt}\frac{\partial^2 x}{\partial J^2} &= -\frac{\partial^3(\varepsilon K_0)}{\partial x^2\partial y}\left(\frac{\partial x}{\partial J}\right)^2-2\frac{\partial^3(\varepsilon K_0)}{\partial x\partial y^2}\frac{\partial x}{\partial J}\frac{\partial y}{\partial J}-\frac{\partial^3(\varepsilon K_0)}{\partial y^3}\left(\frac{\partial y}{\partial J}\right)^2-\frac{\partial^2(\varepsilon K_0)}{\partial x\partial y}\frac{\partial^2 x}{\partial J^2}-\frac{\partial^2(\varepsilon K_0)}{\partial y^2}\frac{\partial^2 y}{\partial J^2} \\ \frac{d}{dt}\frac{\partial^2 y}{\partial J^2} &= \frac{\partial^3(\varepsilon K_0)}{\partial x^3}\left(\frac{\partial x}{\partial J}\right)^2+2\frac{\partial^3(\varepsilon K_0)}{\partial x^2\partial y}\frac{\partial x}{\partial J}\frac{\partial y}{\partial J}+\frac{\partial^3(\varepsilon K_0)}{\partial x\partial y^2}\left(\frac{\partial y}{\partial J}\right)^2+\frac{\partial^2(\varepsilon K_0)}{\partial x^2}\frac{\partial^2 x}{\partial J^2}+\frac{\partial^2(\varepsilon K_0)}{\partial x\partial y}\frac{\partial^2 y}{\partial J^2},\end{aligned} \tag{40}$$

and (29) over Z

$$
\begin{aligned}
\frac{d}{dt}\frac{\partial^2 x}{\partial Z^2} &= -\frac{\partial^3 \mathcal{K}_0}{\partial x^2 \partial y}\left(\frac{\partial x}{\partial Z}\right)^2 - 2\frac{\partial^3 \mathcal{K}_0}{\partial x \partial y^2}\frac{\partial x}{\partial Z}\frac{\partial y}{\partial Z} - \frac{\partial^3 \mathcal{K}_0}{\partial y^3}\left(\frac{\partial y}{\partial Z}\right)^2 - 2\frac{\partial^3 \mathcal{K}_0}{\partial x \partial y \partial P}\frac{\partial x}{\partial Z} - \\
&\quad -2\frac{\partial^3 \mathcal{K}_0}{\partial y^2 \partial P}\frac{\partial y}{\partial Z} - \frac{\partial^2 \mathcal{K}_0}{\partial x \partial y}\frac{\partial^2 x}{\partial Z^2} - \frac{\partial^2 \mathcal{K}_0}{\partial y^2}\frac{\partial^2 y}{\partial Z^2} - \frac{\partial^3 \mathcal{K}_0}{\partial y \partial P^2} \\
\frac{d}{dt}\frac{\partial^2 y}{\partial Z^2} &= \frac{\partial^3 \mathcal{K}_0}{\partial x^3}\left(\frac{\partial x}{\partial Z}\right)^2 + 2\frac{\partial^3 \mathcal{K}_0}{\partial x^2 \partial y}\frac{\partial x}{\partial Z}\frac{\partial y}{\partial Z} + \frac{\partial^3 \mathcal{K}_0}{\partial x \partial y^2}\left(\frac{\partial y}{\partial Z}\right)^2 + 2\frac{\partial^3 \mathcal{K}_0}{\partial x^2 \partial P}\frac{\partial x}{\partial Z} + \\
&\quad +2\frac{\partial^3 \mathcal{K}_0}{\partial x \partial y \partial P}\frac{\partial y}{\partial Z} + \frac{\partial^2 \mathcal{K}_0}{\partial x^2}\frac{\partial^2 x}{\partial Z^2} + \frac{\partial^2 \mathcal{K}_0}{\partial x \partial y}\frac{\partial^2 y}{\partial Z^2} + \frac{\partial^3 \mathcal{K}_0}{\partial x \partial P^2},
\end{aligned} \tag{41}
$$

while mixed derivatives are obtained either by differentiating homogeneous variational equations (28) over Z, or non homogeneous Eqs. (29) over J

$$
\begin{aligned}
\frac{d}{dt}\frac{\partial^2 x}{\partial J \partial Z} &= -\frac{\partial^2 \mathcal{K}_0}{\partial y^2}\frac{\partial^2 y}{\partial J \partial Z} - \frac{\partial^2 \mathcal{K}_0}{\partial x \partial y}\frac{\partial^2 x}{\partial J \partial Z} - \left(\frac{\partial^3 \mathcal{K}_0}{\partial y^2 \partial P} + \frac{\partial^3 \mathcal{K}_0}{\partial y^3}\frac{\partial y}{\partial Z} + \frac{\partial^3 \mathcal{K}_0}{\partial x \partial y^2}\frac{\partial x}{\partial Z}\right)\frac{\partial y}{\partial J} - \\
&\quad - \left(\frac{\partial^3 \mathcal{K}_0}{\partial x \partial y \partial P} + \frac{\partial^3 \mathcal{K}_0}{\partial x \partial y^2}\frac{\partial y}{\partial Z} + \frac{\partial^3 \mathcal{K}_0}{\partial x^2 \partial y}\frac{\partial x}{\partial Z}\right)\frac{\partial x}{\partial J} \\
\frac{d}{dt}\frac{\partial^2 y}{\partial J \partial Z} &= \frac{\partial^2 \mathcal{K}_0}{\partial x \partial y}\frac{\partial^2 y}{\partial J \partial Z} + \frac{\partial^2 \mathcal{K}_0}{\partial x^2}\frac{\partial^2 x}{\partial J \partial Z} + \left(\frac{\partial^3 \mathcal{K}_0}{\partial x \partial y \partial P} + \frac{\partial^3 \mathcal{K}_0}{\partial x \partial y^2}\frac{\partial y}{\partial Z} + \frac{\partial^3 \mathcal{K}_0}{\partial x^2 \partial y}\frac{\partial x}{\partial Z}\right)\frac{\partial y}{\partial J} + \\
&\quad + \left(\frac{\partial^3 \mathcal{K}_0}{\partial x^2 \partial P} + \frac{\partial^3 \mathcal{K}_0}{\partial x^2 \partial y}\frac{\partial y}{\partial Z} + \frac{\partial^3 \mathcal{K}_0}{\partial x^3}\frac{\partial x}{\partial Z}\right)\frac{\partial x}{\partial J},
\end{aligned} \tag{42}
$$

with the initial condition $(0, 0)$.

Table 1. Proper semimajor axis (a_P), proper eccentricity (e_P) and sine of proper inclination ($\sin i_P$), actions Λ, J, Z, frequencies ω_1, ω_2 i ω_3 and derivatives of the frequencies over actions up to the second order for the asteroid (158) Koronis.

Proper elements			Frequencies		
a_P(AU)	e_P	$\sin i_P$	ω_1	$\omega_2[10^{-2}]$	$\omega_3[10^{-3}]$
2.86879	0.0452	0.0375	2.4410096	0.17698266	−0.58617676
Actions			Derivatives of the frequencies		
Λ	$J[10^{-3}]$	$Z[10^{-2}]$			
0.74257530	0.52347921	0.12807165	−0.98810484D+01	−0.61957684D−01	−0.17779001D−04
			−0.61957684D−01	0.49731945D−03	−0.25149875D−04
			−0.17779001D−04	−0.25149875D−04	0.94192010D−02
			Derivatives of the second order		
			0.51488061D+02	0.56724344D+01	−0.46119948D+02
			0.56724344D+01	0.17067368D−01	0.39393382D+00
			−0.46119948D+02	0.39393382D+00	0.11253108D+04
			0.56724344D+01	0.17067368D−01	0.39393382D+00
			0.17067368D−01	0.34914856D+01	0.33231819D+00
			0.39393382D+00	0.33231819D+00	−0.26951264D+01
			−0.46119948D+02	0.39393382D+00	0.11253108D+04
			0.39393382D+00	0.33231819D+00	−0.26951264D+01
			0.11253108D+04	−0.26951264D+01	0.11762035D+04

4. NUMERICAL EXAMPLE

To check the derived equations, as well as the computer code[1] written in FORTRAN, in Table 1 we give a numerical example for the asteroid (158) Koronis with the values of derivatives calculated from Eqs. (13) – (42). Apart from the gravitational constant and the mass of the Sun, it turned out to be convenient to set to unity also the semimajor axis of Jupiter. Thus, the frequencies in Table 1 are given in units of Jupiter's mean motion ($n_J \approx 300$ arcsec/day). Integrals from the above expressions are computed numerically by dividing the period T in 200 equidistant points, thus achieving the required accuracy while still not consuming too much computer time.

Acknowledgements – I thank Dr Z. Knežević for many useful comments. Dr A. Lemaitre has kindly provided code to calculate proper elements for highly inclined asteroidal orbits. I am also grateful to B. Novaković who checked some of the Eqs. (20) – (42). This research has been supported by the Ministry of Science and Environmental Protection of the Republic of Serbia (Project No 146004 "Dynamics of Celestial Bodies, Systems and Populations")

REFERENCES

Guzzo, M., Knežević, Z., Milani, A.: 2002, *Celest. Mech. Dyn. Astron.*, **83**, 121.
Henrard, J.: 1990, *Celest. Mech. Dyn. Astron.*, **49**, 43.
Henrard, J. and Lemaitre, A.: 1986, *Celest. Mech. Dyn. Astron.*, **39**, 213.
Kozai, Y.: 1962, *Astron. J.*, **67**, 591.
Lemaitre, A. and Morbidelli, A.: 1994, *Celest. Mech. Dyn. Astron.*, **60**, 29.
Morbidelli, A.: 2002, Modern Celestial Mechanics; Aspects of Solar System Dynamics, Taylor & Francis/Cambridge Scientific Publishers.
Morbidelli, A. and Henrard, J.: 1991, *Celest. Mech. Dyn. Astron.*, **51**, 131.
Nekhoroshev, N.N.: 1977, *Uspekhi. Mat. Nauk*, **32**, 5.
Pavlović, R. and Guzzo, M.: 2007, *Mon. Not. R. Astron. Soc.*, submitted.

ТРЕЋИ ИЗВОДИ ИНТЕГРАБИЛНОГ ДЕЛА ХАМИЛТОНИЈАНА АСТЕРОИДА

R. Pavlović

Astronomical Observatory, Volgina 7, 11160 Beograd 74, Serbia

УДК 521.19 : 523.44
Оригинални научни рад

Пре примене теореме Нехорошева (1977) на астероиде треба проверити да ли је испуњен потребан геометријски услов: конвексност, квазиконвексност или 3–џет недегенерисаност. За проверу ових услова неопходно је израчунати изводе до трећег реда интеграбилног дела Хамилтонијана и испитати њихове особине. Овај чланак даје експлицитне изразе за изводе до трећег реда интеграбилног дела Хамилтонијана астероида по акцијама.

[1]Input values for the code are the proper elements of asteroids, which can be found at AstDyS site `http://hamilton.dm.unipi.it/cgi-bin/astdys/astibo`.