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R FLUIDS

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SUMMARY: A theory of collisionless fluids is developed in a unified picture, where nonrotating $(\widetilde{\Omega_1} = \widetilde{\Omega_2} = \widetilde{\Omega_3} = 0)$ figures with some given random velocity component distributions, and rotating $(\widetilde{\Omega_1} \neq \widetilde{\Omega_2} \neq \widetilde{\Omega_3})$ figures with a different random velocity component distributions, make adjoint configurations to the same system. R fluids are defined as ideal, self-gravitating fluids satisfying the virial theorem assumptions, in presence of systematic rotation around each of the principal axes of inertia. To this aim, mean and rms angular velocities and mean and rms tangential velocity components are expressed, by weighting on the moment of inertia and the mass, respectively. The figure rotation is defined as the mean angular velocity, weighted on the moment of inertia, with respect to a selected axis. The generalized tensor virial equations (Caimmi and Marmo 2005) are formulated for R fluids and further attention is devoted to axisymmetric configurations where, for selected coordinate axes, a variation in figure rotation has to be counterbalanced by a variation in anisotropy excess and vice versa. A microscopical analysis of systematic and random motions is performed under a few general hypotheses, by reversing the sign of tangential or axial velocity components of an assigned fraction of particles, leaving the distribution function and other parameters unchanged (Meza 2002). The application of the reversion process to tangential velocity components is found to imply the conversion of random motion rotation kinetic energy into systematic motion rotation kinetic energy. The application of the reversion process to axial velocity components is found to imply the conversion of random motion translation kinetic energy into systematic motion translation kinetic energy, and the loss related to a change of reference frame is expressed in terms of systematic motion (imaginary) rotation kinetic energy. A number of special situations are investigated in greater detail. It is found that an R fluid always admits an adjoint configuration where figure rotation occurs around only one principal axis of inertia (R3 fluid), which implies that all the results related to R3 fluids (Caimmi 2007) may be extended to R fluids. Finally, a procedure is sketched for deriving the spin parameter distribution (including imaginary rotation) from a sample of observed or simulated large-scale collisionless fluids i.e. galaxies and galaxy clusters.

Key words. Galaxies: clusters: general – Galaxies: halos

1. INTRODUCTION

Due to particle shocks, collisional fluids (e.g. stars, gas clouds) exhibit an isotropic stress tensor $(\sigma_{11}^2 = \sigma_{22}^2 = \sigma_{33}^2)$, where σ_{pp}^2 is the rms random velocity component on the axis x_p. The absence of particle shocks (leaving aside extreme situations, such as high-density galactic nuclei) makes a different situation in collisionless fluids (e.g. galaxies, galaxy clusters), where the stress tensor is - in general - anisotropic ($\sigma_{11}^2 \neq \sigma_{22}^2 \neq \sigma_{33}^2$). The shape of the body is determined by systematic rotation, which is quantified by a spin parameter (null for nonrotating configurations), and/or by the difference between stress tensor diagonal components, or any equivalent anisotropy indicator, related to the rotation and an equatorial principal axis of inertia, respectively (null for configurations where the random velocity component distribution is isotropic). A description of collisionless fluids based on the equivalence of systematic and random motions with respect to the shape, appears to be highly rewarding and it would provide further insight on the properties of stellar and galaxy systems.

In an earlier attempt (Caimmi 1996), the stress tensor was expressed as the sum of two terms, one related to a random (isotropic) velocity component distribution, and the other to anisotropic internal motions within the system. Further investigation was devoted to the simplest situation, where the system is made of two equal components rotating at the same rate but in opposite sense. Then it has been recognized that the anisotropy excess may be related to real rotation, if the shape is flattened, and to imaginary rotation, if the shape is elongated, with respect to the rotation axis.

A later approach (Caimmi and Marmo 2005) has been restricted to homeoidally striated density profiles, for which the tensor virial equations were formulated and generalized to unrelaxed configurations. The kinetic-energy tensor has been expressed as the sum of two terms, one related to systematic rotation obeying an assigned law, and the other to the remaining motions, e.g. random motions, streaming motions, radial motions. Finally, an expression of the spin parameter in terms of the anisotropy excess showed the role of systematic and remaining motions in flattening or elongating the shape.

The above results were improved and extended in subsequent work (Caimmi 2006, hereafter quoted as C06), where the imaginary rotation was related to negative anisotropy excess. The sequences of configurations for which the generalized tensor virial equations hold have been determined for homeoidally striated Jacobi ellipsoids including prolate shapes induced by imaginary rotation. The results of numerical simulations on the stability of rapidly rotating spherical configurations (Meza 2002) were interpreted in the light of the theory. To this respect, the key argument is that the reversion (from clockwise to counterclockwise or vice versa) of tangential velocity components related to an assigned fraction of partices, preserves the potential energy, the kinetic energy, and the distribution function (Lynden-Bell 1960, 1962, Meza 2002).

The study of homeoidally striated Jacobi ellipsoids was extended to a more general class of bodies (R3 fluids) in a recent paper (Caimmi 2007, hereafter quoted as C07), where the contribution of radial and tangential velocity components in the equatorial plane was investigated in more detail. In addition, mean and rms (weighted on the moment of inertia) angular velocity were defined, and related to systematic and random motion tangential kinetic-energy tensor components, respectively. Also for R3 fluids, was been realized that the effect of (positive or negative) anisotropy excess is equivalent to additional figure (real or imaginary) rotation.

The current attempt is aimed to extend the above mentioned results to a still more general class of bodies, R fluids, defined as ideal, self-gravitating, collisionless fluids where rotation occurs about each of the principal axes of inertia. It will be shown that R fluids always admit an adjoint configuration where figure rotation occurs around a single principal axis, that is a R3 fluid. Accordingly, all the results which hold for R3 fluids may be extended to R fluids.

The paper is organized as follows. A number of basic definitions are provided in Section 2, including the inertia tensor, the angular-velocity tensor, and the angular-momentum tensor. The generalized tensor virial equations for R fluids are formulated in Section 3. The microscopical analysis of systematic and random motions, for a collisionless fluid made of N identical particles, is carried out in Section 4, where a velocity component reversion process is defined, and a number of special situations are analysed in detail with respect to kinetic energy changes from random to systematic motions and vice versa. A procedure aimed at the derivation of the spin parameter distribution (including imaginary rotation) from an assigned sample of observed or simulated objects, is outlined in Section 5. Some concluding remarks are reported in Section 6.

2. ANGULAR-VELOCITY AND ANGULAR-MOMENTUM TENSOR

In the special case of solid bodies, rotation is rigid and occurs around a single axis which, in turn, can remain fixed or change its direction. Accordingly, the angular momentum and the rotation kinetic energy read (e.g. Landau and Lifchitz 1966, Chap. VI, §§ 31-33; hereafter quoted as LL66):

$$J_r = \sum_{s=1}^{3} I'_{rs}\Omega_s, \qquad r = 1, 2, 3 \ , \tag{1}$$

$$T_{rot} = \frac{1}{2}\sum_{r=1}^{3}\sum_{s=1}^{3} I'_{rs}\Omega_r\Omega_s \ , \tag{2}$$

where $\vec{J} = (J_1, J_2, J_3)$ is the angular-momentum vector, $\overrightarrow{\Omega} = (\Omega_1, \Omega_2, \Omega_3)$ the angular-velocity vector, and I' the inertia tensor:

$$I'_{rs} = \int_S \rho(x_1, x_2, x_3)\left[\delta_{rs}\sum_{r=1}^{3} x_r^2 - x_r x_s\right] d^3S \ , \tag{3}$$

related to the density profile, ρ, within the volume, S, δ_{rs} being the Kronecker symbol. The diagonal components of the inertia tensor, I'_{11}, I'_{22}, I'_{33}, are the moments of inertia with respect to the axes, x_1, x_2, x_3, respectively.

In addition, the inertia tensor is symmetric and of second rank, which implies the existence of a reference frame, $(\mathsf{O}'\,X_1\,X_2\,X_3)$, where the inertia tensor is diagonal (LL66, Chap. VI, § 32):

$$I'_{rs} = \delta_{rs} I'_{rr} \qquad (4)$$

the coordinate axes of this reference frame coincide with the principal axes of inertia, and the diagonal components define the principal moments of inertia. Accordingly, Eqs. (1) and (2) reduce to:

$$J_r = I'_{rr}\Omega_r \quad , \qquad r = 1,2,3 \quad , \qquad (5)$$

$$T_{rot} = \frac{1}{2}\left(I'_{11}\Omega_1^2 + I'_{22}\Omega_2^2 + I'_{33}\Omega_3^2\right) \quad , \qquad (6)$$

where, in addition (LL66, Chap. VI, § 32):

$$I'_{rr} \le I'_{ss} + I'_{tt} \quad , \quad r = 1,2,3 \quad , \quad s = 2,3,1 \quad , \quad t = 3,1,2 \quad . \qquad (7)$$

The inertia tensor has been defined in a different way, as (e.g. Chandrasekhar 1969, Chap. 2, § 9; Binney and Tremaine 1987, Chap. 4, § 3):

$$I_{rs} = \int_S \rho(x_1,x_2,x_3) x_r x_s \, d^3S \quad , \qquad (8)$$

and the combination of Eqs. (3) and (8) yields:

$$I'_{rs} = \delta_{rs}\sum_{r=1}^{3} I_{rr} - I_{rs} \quad , \qquad (9)$$

$$I'_{rr} = I_{ss} + I_{tt} \quad , \qquad r \ne s \ne t \quad , \qquad (10)$$

$$I'_{rs} = -I_{rs} \quad , \qquad r \ne s \quad , \qquad (11)$$

or:

$$2I_{rr} = I'_{ss} + I'_{tt} - I'_{rr} \quad , \qquad r \ne s \ne t \quad , \qquad (12)$$

$$I_{rs} = -I'_{rs} \quad , \qquad r \ne s \quad , \qquad (13)$$

which translates one formulation into the other (e.g. Bett et al. 2007).

In the general case of (collisional or collisionless) fluids, rotation could be different from that of the solid-body, and around each principal axis of inertia. Let $(\mathsf{O}\,x_1\,x_2\,x_3)$ be a generic reference frame and $(\mathsf{O}'\,X_1\,X_2\,X_3)$ a reference frame where the origin coincides with the centre of inertia, and the coordinate axes coincide with the principal axes of inertia. Let the coordinate axes, X_1, X_2, X_3, be defined as the principal axes. Let $\overrightarrow{\Omega_1}$, $\overrightarrow{\Omega_2}$, $\overrightarrow{\Omega_3}$, be the angular-velocity vectors (to be specified later) related to the principal axes of inertia. Let Ω_{rs} be the component of the vector $\overrightarrow{\Omega_r}$ on the coordinate axes x_s. The (3×3) tensor, Ω_{rs}, is defined as the angular-velocity tensor of the system under consideration, with respect to the reference frame, $(\mathsf{O}\,x_1\,x_2\,x_3)$. Let $\overrightarrow{\omega_1}$, $\overrightarrow{\omega_2}$, $\overrightarrow{\omega_3}$, be the angular-velocity vectors related to the coordinate axes, x_1, x_2, x_3. The following relation holds:

$$\omega_s = \sum_{r=1}^{3}\Omega_{rs} \quad , \qquad s = 1,2,3 \quad , \qquad (14)$$

and the angular-velocity tensor, $\omega_{rs} = \Omega_{rs}$, can formally be defined. Similarly, the (3×3) angular-momentum tensor is expressed as:

$$J'_{rs} = I'_{rs}\omega_{rs} \quad , \qquad (15)$$

$$J'_s = \sum_{r=1}^{3} I'_{rs}\omega_{rs} \quad , \qquad (16)$$

where the inertia tensor is related to the reference frame $(\mathsf{O}\,x_1\,x_2\,x_3)$.

In the special case where the reference frame, $(\mathsf{O}\,x_1\,x_2\,x_3)$, coincides with $(\mathsf{O}'\,X_1\,X_2\,X_3)$, one has $\Omega_{rs} = \delta_{rs}\Omega_{rr}$. Accordingly, Eqs. (14)-(16) reduce to:

$$\omega_s = \Omega_s = \Omega_{ss} = \omega_{ss} \quad , \qquad (17)$$

$$J'_{rs} = \delta_{rs} I'_{rr}\Omega_{rr} \quad , \qquad (18)$$

$$J'_s = J_s = I'_{ss}\Omega_s \quad , \qquad (19)$$

where $I'_{tt} = I_{rr} + I_{ss}$, $r \ne s \ne t$, represents the moment of inertia with respect to the principal axis of inertia, x_t. From this point on, it shall be assumed that the origin coincides with the centre of inertia, and the coordinate axes coincide with the principal axes of inertia.

The rotation kinetic-energy tensor is defined as:

$$(T_{rot})_{rs} = \frac{1}{2} I'_{rs}\Omega_r\Omega_s = \frac{1}{2}\delta_{rs} I'_{rr}\Omega_r^2 \quad , \qquad (20)$$

where the diagonal components of the angular-velocity tensor are expressed as (C07):

$$\Omega_r = \widetilde{\Omega_r} = \frac{1}{I'_{rr}}\int_S \left|\overrightarrow{\Omega_r}(x_1,x_2,x_3,t)\right| w_r^2 \rho(x_1,x_2,x_3,t) \times d^3S, \quad r \ne s \ne t, \qquad (21)$$

$$\left|\overrightarrow{\Omega_r}(x_1,x_2,x_3,t)\right| = \frac{v_{\phi_r}(x_1,x_2,x_3,t)}{w_r} \quad , \qquad (22)$$

$$w_r = (x_s^2 + x_t^2)^{1/2} \quad , \qquad (23)$$

and $\Omega_r(x_1,x_2,x_3,t)$ is the mean value related to all the particles at the time, t, within the infinitesimal volume element, $d^3S = dx_1\,dx_2\,dx_3$, centred on the point, $\mathsf{P}(x_1,x_2,x_3)$, v_{ϕ_r} is the tangential velocity component in the $(\mathsf{O}\,x_s\,x_t)$ principal plane, and the moment of inertia, $I'_{rr} = I_{ss} + I_{tt}$, reads:

$$I'_{rr} = \int_S w_r^2 \rho(x_1,x_2,x_3,t)\, d^3S \quad , \qquad r \ne s \ne t \quad , \qquad (24)$$

as expected from the theorem of the mean, in connection with Eq. (21).

Similarly, the mean square diagonal components of the angular-velocity tensor are expressed as:

$$\Omega_r^2 = \widetilde{(\Omega_r^2)} = \frac{1}{I'_{rr}}\int_S \left|\overrightarrow{\Omega_r}(x_1,x_2,x_3,t)\right|^2 w_r^2 \times \rho(x_1,x_2,x_3,t)\, d^3S, \; r \ne s \ne t, \qquad (25)$$

and the related variance reads:

$$\left(\sigma_{\widetilde{\Omega_r}\widetilde{\Omega_r}}\right)^2 = \widetilde{(\Omega_r^2)} - (\widetilde{\Omega_r})^2 \quad . \tag{26}$$

At this stage, it may be useful to extend and generalize the definition of figure rotation.

Figure rotation. *Given an R fluid, the figure rotation is defined as the mean angular velocity, weighted on the moment of inertia, with respect to a selected principal axis.*

In terms of tangential velocity components, the counterparts of Eqs. (21), (25), and (26) read:

$$\widetilde{v_{\phi_r}} = \frac{1}{MR_{Gr}} \int_S \left|\overrightarrow{\Omega_r}(x_1, x_2, x_3, t)\right| w_r^2 \rho(x_1, x_2, x_3, t) \times d^3S \quad ; \quad r \neq s \neq t \quad , \tag{27}$$

$$\widetilde{(v_{\phi_r}^2)} = \frac{1}{M} \int_S \left|\overrightarrow{\Omega_r}(x_1, x_2, x_3, t)\right|^2 w_r^2 \rho(x_1, x_2, x_3, t) \times d^3S; \quad r \neq s \neq t, \tag{28}$$

$$\left(\sigma_{\widetilde{v_{\phi_r}}\widetilde{v_{\phi_r}}}\right)^2 = \widetilde{(v_{\phi_r}^2)} - (\widetilde{v_{\phi_r}})^2 \quad , \tag{29}$$

where R_{Gr} is the curl radius with respect to the principal axis x_r:

$$R_{Gr} = \left(\frac{I'_{rr}}{M}\right)^{1/2} \quad , \tag{30}$$

and the combination of Eqs. (25) and (28); (21) and (27); (26) and (29) yields:

$$M\widetilde{(v_{\phi_r}^2)} = I'_{rr}\widetilde{(\Omega_r^2)} \quad ; \tag{31a}$$

$$M(\widetilde{v_{\phi_r}})^2 = I'_{rr}(\widetilde{\Omega_r})^2 \quad ; \tag{31b}$$

$$M\left(\sigma_{\widetilde{\phi_r}\widetilde{\phi_r}}\right)^2 = I'_{rr}\left(\sigma_{\widetilde{\Omega_r}\widetilde{\Omega_r}}\right)^2 \quad ; \tag{31c}$$

which relate tangential velocity components in the $(\mathsf{O}\, x_s\, x_t)$ principal plane, to angular velocity components with respect to the principal axis x_r.

3. THE GENERALIZED TENSOR VIRIAL EQUATIONS FOR R FLUIDS

Let R fluids be defined as (collisional or collisionless) ideal self-gravitating fluids where figure rotation occurs around all the three principal axes of inertia. Let $(\mathsf{O}\, x_1\, x_2\, x_3)$ be a reference frame where the origin coincides with the centre of inertia, and the coordinate axes coincide with the principal axes of inertia. Then the mean radial velocity components must necessarily equal zero:

$$\overline{v_{w_r}} = 0 \quad , \qquad \overline{(v_{w_r}^2)} = (\sigma_{w_r w_r})^2 \quad , \tag{32}$$

where v_{w_r} is the radial velocity component on the principal plane $(\mathsf{O}\, x_s\, x_t)$, perpendicular to the principal axis x_r. Let positive and negative radial velocity components be defined as directed outwards and inwards, respectively. The same holds for the mean tangential velocity components:

$$\overline{v_{\phi_r}} = 0 \quad . \qquad \overline{(v_{\phi_r}^2)} = (\sigma_{\phi_r\phi_r})^2 \quad , \tag{33}$$

even in presence of systematic rotation. Let positive and negative tangential velocity components be defined as rotating counterclockwise and clockwise, respectively.

The kinetic-energy tensor may be expressed as the sum of two contributions: one, related to systematic motions, and the other, related to random motions (C07). The result is:

$$T_{k_s k_t} = (T_{\rm sys})_{k_s k_t} + (T_{\rm rdm})_{k_s k_t} \quad ; \qquad k = w, \phi \quad ; \tag{34}$$

where the terms on the right-hand side, using Eqs. (31) and (32), can be expressed as:

$$(T_{\rm sys})_{w_s w_t} = 0 \quad , \tag{35a}$$

$$(T_{\rm rdm})_{w_s w_t} = \frac{1}{2}\delta_{st} M \left(\sigma_{w_s w_s}\right)^2 \quad , \tag{35b}$$

$$(T_{\rm sys})_{\phi_s\phi_t} = \frac{1}{2}\delta_{st} I'_{ss} (\widetilde{\Omega_s})^2 \quad , \tag{36a}$$

$$(T_{\rm rdm})_{\phi_s\phi_t} = \frac{1}{2}\delta_{st} I'_{ss} \left(\sigma_{\widetilde{\Omega_s}\widetilde{\Omega_s}}\right)^2 \quad . \tag{36b}$$

Keeping in mind that nondiagonal components are null in the case under discussion, only diagonal components shall be considered from this point on. The combination of Eqs. (26) and (36) yields:

$$T_{\phi_r\phi_r} = \frac{1}{2} I'_{rr} \widetilde{(\Omega_r^2)} \quad , \tag{37}$$

which depends on the density profile via the moment of inertia, I'_{rr}, and the tangential velocity component distribution via the mean square angular velocity, $\widetilde{(\Omega_r^2)}$, regardless of the fraction of systematic and random motions.

In terms of the contributions related to the axial components of the kinetic-energy tensor, T_{ss} and T_{tt}, Eqs. (34), (36), and (37) read:

$$(T_{\phi_r\phi_r})_{\ell\ell} = \frac{1}{2} I_{\ell\ell} \widetilde{(\Omega_r^2)} \quad , \qquad \ell = s, t \quad , \tag{38a}$$

$$[(T_{\rm sys})_{\phi_r\phi_r}]_{\ell\ell} = \frac{1}{2} I_{\ell\ell} (\widetilde{\Omega_r})^2 \quad , \qquad \ell = s, t \quad , \tag{38b}$$

$$[(T_{\rm rdm})_{\phi_r\phi_r}]_{\ell\ell} = \frac{1}{2} I_{\ell\ell} \left[\widetilde{(\Omega_r^2)} - (\widetilde{\Omega_r})^2\right], \ \ell = s, t, \tag{38c}$$

where Eq. (10) has been used.

The invariance of a vector with respect to a change of the reference frame, implies the validity of the relations (C07):

$$\overline{(v_{w_r}^2)} + \overline{(v_{\phi_r}^2)} = \overline{(v_s^2)} + \overline{(v_t^2)} \quad , \tag{39}$$

$$(\overline{v_{w_r}})^2 + (\overline{v_{\phi_r}})^2 = (\overline{v_s})^2 + (\overline{v_t})^2 \quad , \tag{40}$$

$$(\sigma_{w_r w_r})^2 + (\sigma_{\phi_r\phi_r})^2 = (\sigma_{ss})^2 + (\sigma_{tt})^2 \quad , \tag{41}$$

where the velocity components on the x_s and x_t principal axes are labelled by the indices, s and t, respectively.

The combination of Eqs. (26), (29), and (39)-(41) yields:

$$(\sigma_{w_r w_r})^2 = (\sigma_{ss})^2 + (\sigma_{tt})^2 - \frac{I'_{rr}}{M}\left[(\widetilde{\Omega_r^2}) - (\widetilde{\Omega_r})^2\right] \quad , \tag{42}$$

which makes Eqs. (34) and (35) translate into:

$$(T_{w_r w_r})_{\ell\ell} = [(T_{\rm rdm})_{w_r w_r}]_{\ell\ell}$$
$$= \frac{1}{2}M\sigma_{\ell\ell}^2 - \frac{1}{2}I_{\ell\ell}\left[(\widetilde{\Omega_r^2}) - (\widetilde{\Omega_r})^2\right] \quad , \quad \ell = s, t \quad , \tag{43a}$$
$$[(T_{\rm sys})_{w_r w_r}]_{\ell\ell} = 0 \quad , \qquad \ell = s, t \quad , \tag{43b}$$

in terms of the contributions related to the axial components of the kinetic-energy tensor, T_{ss} and T_{tt}.

The generalized tensor virial equations of the second order can be formulated, extending the procedure used for R3 fluids (C07). The result is:

$$I_{rr}\left[(\widetilde{\Omega_s})^2 + (\widetilde{\Omega_t})^2\right] + M\zeta_{rr}\sigma^2 + (E_{\rm pot})_{rr} = 0 \ , \tag{44}$$
$$\sigma^2 = \sigma_{11}^2 + \sigma_{22}^2 + \sigma_{33}^2 \quad , \tag{45}$$

$$\zeta_{pp} = \frac{(\widetilde{T}_{\rm rdm})_{pp}}{T_{\rm rdm}} = \frac{\sigma_{pp}^2}{\sigma^2} \quad , \qquad p = 1, 2, 3 \quad , \tag{46a}$$
$$\zeta_{11} + \zeta_{22} + \zeta_{33} = \frac{\widetilde{T}_{\rm rdm}}{T_{\rm rdm}} = \frac{\widetilde{\sigma}^2}{\sigma^2} = \zeta \quad , \tag{46b}$$

where $\widetilde{\Omega_s}$ and $\widetilde{\Omega_t}$ are mean angular velocity components due to systematic rotation around x_s and x_t axes, respectively, $(E_{\rm pot})_{rr}$ is the self potential-energy tensor, ζ_{rr} may be conceived as generalized anisotropy parameters (Caimmi and Marmo 2005, C06, C07), and $\widetilde{T}_{\rm rdm}$ is the effective random kinetic energy i.e. the right amount needed for an instantaneous configuration to satisfy the usual tensor virial equations of the second order, defined by the effective anisotropy parameters (C07):

$$\widetilde{\zeta}_{pp} = \frac{(\widetilde{T}_{\rm rdm})_{pp}}{\widetilde{T}_{\rm rdm}} = \frac{\zeta_{pp}}{\zeta} \quad , \qquad p = 1, 2, 3 \quad ; \tag{47a}$$
$$\widetilde{\zeta}_{11} + \widetilde{\zeta}_{22} + \widetilde{\zeta}_{33} = 1 \quad , \tag{47b}$$

and the condition, $\zeta = 1$, or $\zeta_{pp} = \widetilde{\zeta}_{pp}$, $p = 1, 2, 3$, reduces Eqs. (44) to their standard counterparts. To get further insight, a microscopical analysis is needed.

In the special case of axisymmetric configurations, $I_{11} = I_{22}$, $(E_{\rm pot})_{11} = (E_{\rm pot})_{22}$, and the combination of the related tensor virial equations, expressed by Eq. (44), yields:

$$I_{pp}\left[(\widetilde{\Omega_q})^2 - (\widetilde{\Omega_p})^2\right] = M\sigma^2(\zeta_{qq} - \zeta_{pp}) \quad ;$$
$$p = 1, 2 \quad ; \quad q = 2, 1 \quad ; \tag{48}$$

where the figure rotation excess, $[(\widetilde{\Omega_q})^2 - (\widetilde{\Omega_p})^2]$, is counterbalanced by an anisotropy excess, $(\zeta_{qq} - \zeta_{pp})$; in particular, a null figure rotation excess implies a null anisotropy excess and vice versa. Accordingly, a flattening in the $(\mathsf{O}\, x_p\, x_r)$ principal plane, induced by the figure rotation excess, has to be counterbalanced by an elongation on the x_q principal axis, induced by the anisotropy excess, to yield an axisymmetric configuration with respect to the x_r principal axis.

4. MICROSCOPICAL ANALYSIS OF SYSTEMATIC AND RANDOM MOTIONS

Consider a collisionless R fluid. Let N be the total number of particles and m the mean particle mass in absence of mass segregation i.e. let local and global mean particle masses coincide. For simplicity, the equivalent description (C07) involving N identical particles of mass, m, shall be considered. Let v_{ϕ_r} be the tangential velocity component in $(\mathsf{O}\, x_s\, x_t)$ principal plane. It is worth noting (e.g. Meza 2002) that the distribution function is independent of the sign of v_{ϕ_r}, and the whole set of possible configurations is characterized by an equal amount of both kinetic and potential energies. Numerical simulations show that spherical systems, even if rapidly rotating, are dynamically stable after reversion of the tangential velocity component in an assigned fraction of particles (Meza 2002).

For the sake of simplicity, let the initial configuration be nonrotating ($\overline{v_{\phi_r}} = 0$) and with isotropic random velocity component distribution ($\zeta_{11} = \zeta_{22} = \zeta_{33}$). In the case under discussion of identical particles, $m^{(i)} = m$, $1 \le i \le N$, the centre of inertia velocity components, $v_{{\rm C}r}$, equal the corresponding arithmetic means:

$$v_{{\rm C}r} = \frac{\sum_{i=1}^{N} m^{(i)} v_r^{(i)}}{\sum_{i=1}^{N} m^{(i)}} = \frac{m\sum_{i=1}^{N} v_r^{(i)}}{Nm} = \overline{v_r} \quad , \tag{49}$$

and the moments of inertia, I'_{rr}, reduce to:

$$I'_{rr} = \sum_{i=1}^{N} m^{(i)}\left[w_r^{(i)}\right]^2 = m\sum_{i=1}^{N}\left[w_r^{(i)}\right]^2 = MR_{Gr}^2 \quad , \tag{50}$$

according to Eq. (30).

The weighted mean, mean square, and rms tangential velocity components, expressed by Eqs. (27)-(29), read:

$$\widetilde{v_{\phi_r}} = \frac{1}{M}\sum_{i=1}^{N} m^{(i)} v_{\phi_r}^{(i)} = \frac{m}{M}\sum_{i=1}^{N} v_{\phi_r}^{(i)} = \overline{v_{\phi_r}} \quad , \tag{51}$$

$$\widetilde{(v^2_{\phi_r})} = \frac{1}{M}\sum_{i=1}^{N} m^{(i)}\left[v^{(i)}_{\phi_r}\right]^2 = \frac{m}{M}\sum_{i=1}^{N}\left[v^{(i)}_{\phi_r}\right]^2 = \overline{(v^2_{\phi_r})}, \tag{52}$$

$$\left(\sigma_{\widetilde{\phi_r}\widetilde{\phi_r}}\right)^2 = \overline{(v^2_{\phi_r})} - (\overline{v_{\phi_r}})^2 = (\sigma_{\phi_r\phi_r})^2 \quad , \tag{53}$$

and Eqs. (31) reduce to:

$$\overline{(v^2_{\phi_r})} = R^2_{Gr}\widetilde{(\Omega^2_r)} \quad ; \tag{54a}$$

$$(\overline{v_{\phi_r}})^2 = R^2_{Gr}(\widetilde{\Omega_r})^2 \quad ; \tag{54b}$$

$$(\sigma_{\phi_r\phi_r})^2 = R^2_{Gr}\left(\sigma_{\widetilde{\Omega_r}\widetilde{\Omega_r}}\right)^2 \quad ; \tag{54c}$$

which relate weighted angular velocities with respect to the x_r axis to mean tangential velocity components on the $(\mathsf{O}\, x_s\, x_t)$ principal plane.

At this stage, let the tangential velocity component of a fraction, n/N, of particles, be reversed in equal sense (from clockwise to counterclockwise or vice versa), according to the following assumptions:

(i) Both the number, n, of particles where the tangential velocity component was reversed, and the number, $N - n$, of particles which remain unchanged, are sufficiently large, $1 << n << N$, $0 \le n \le \mathrm{Int}(N/2)$.

(ii) The fraction, n_k/N_k, of particles where the tangential velocity component has been reversed, within a generic volume element, S_k, is independent of the volume element, $n_k/N_k = n/N$.

(iii) The system is made of identical particles, $m^{(i)} = m$, $M = mN$.

(iv) After tangential velocity components have been reversed in n_k particles within a generic volume element, S_k, on a total of N_k, a second set of n_k particles (among the remaining $N_k - n_k$) exists, where the tangential velocity component of any particle equals its counterpart belonging to the first set.

In the following, the above process shall be quoted as "the reversion process".

Obviously, mean square tangential velocity components, $\overline{(v^2_{\phi_r})}$, are left unchanged by the reversion process. On the contrary, mean tangential velocity components after the reversion process read:

$$\overline{v_{\phi_r}} = \frac{1}{N}\sum_{i=1}^{N} v^{(i)}_{\phi_r} = \frac{1}{N}\left[\sum_{i=1}^{2n} v^{(i)}_{\phi_r} + \sum_{i=2n+1}^{N} v^{(i)}_{\phi_r}\right] \quad , \tag{55}$$

where the first sum within brackets relates to particles where the reversion process has occurred and their counterparts with equal tangential velocity components, while the second sum comprises the remaining particles and necessarily equals the mean tangential velocity component before the occurrence of the reversion process, which is null in the case under discussion. Accordingly, Eq. (55) reduces to:

$$\overline{v_{\phi_r}} = \frac{2n}{N}(\overline{v_{\phi_r}})_n \quad , \tag{56a}$$

$$(\overline{v_{\phi_r}})_n = \frac{1}{2n}\sum_{i=1}^{2n} v^{(i)}_{\phi_r} = \frac{1}{n}\sum_{i=1}^{n} v^{(i)}_{\phi_r} \quad , \tag{56b}$$

keeping in mind that the first sum is performed on couples of particles with equal tangential velocity components.

The validity of Eqs. (55) and (56) still holds if tangential velocity components, v_{ϕ_r}, are replaced by axial velocity components, v_r. The combination of Eqs. (49) and (56a) yields:

$$v_{\mathrm{C}r} = \overline{v_r} = \frac{2n}{N}(\overline{v_r})_n \quad , \tag{57}$$

which is the velocity component of the centre of inertia with respect to the principal axis x_r, after the reversion process.

The total kinetic energy is left unchanged by the reversion process but, a fraction of random motion kinetic energy is turned into systematic motion kinetic energy. In the following, the reversion process shall be discussed in more detail for a number of different situations.

4.1. Tangential velocity component reversion

Performing the reversion process on a given fraction of particles, n/N, with respect to tangential velocity components, implies the conversion of random (rotation) motion kinetic energy into systematic (rotation) motion kinetic energy, as:

$$\Delta(T_{\mathrm{rdm}})_{\phi_r\phi_r} = -\Delta(T_{\mathrm{sys}})_{\phi_r\phi_r} = -\frac{1}{2}M(\overline{v_{\phi_r}})^2 = -\frac{2n}{N}nm[(\overline{v_{\phi_r}})_n]^2 \quad , \tag{58}$$

where the remaining parameters are left unchanged.

The occurrence of the reversion process implies the following energy changes:

$$T_{\mathrm{rdm}} \to T_{\mathrm{rdm}} - \frac{2n}{N}nm[(\overline{v_{\phi_r}})_n]^2 \quad , \tag{59}$$

$$(T_{\mathrm{rdm}})_{\phi_r\phi_r} \to (T_{\mathrm{rdm}})_{\phi_r\phi_r} - \frac{2n}{N}nm[(\overline{v_{\phi_r}})_n]^2 \quad , \tag{60}$$

$$(T_{\mathrm{rdm}})_{\ell\ell} \to (T_{\mathrm{rdm}})_{\ell\ell} - \frac{1}{2}\frac{2n}{N}nm[(\overline{v_{\phi_r}})_n]^2 \quad , \quad \ell = s, t \quad , \tag{61}$$

$$T_{\mathrm{sys}} \to 0 + \frac{2n}{N}nm[(\overline{v_{\phi_r}})_n]^2 \quad , \tag{62}$$

$$(T_{\mathrm{sys}})_{\phi_r\phi_r} \to 0 + \frac{2n}{N}nm[(\overline{v_{\phi_r}})_n]^2 \quad , \tag{63}$$

$$(T_{\mathrm{sys}})_{\ell\ell} \to 0 + \frac{1}{2}\frac{2n}{N}nm(\overline{v_{\phi_r}})^2_n \quad , \qquad \ell = s, t \quad , \tag{64}$$

while the contributions from random radial motions in the equatorial plane, $(T_{\mathrm{rdm}})_{w_r w_r}$, and along the rotation axis, $(T_{\mathrm{rdm}})_{rr}$, remain unchanged.

With the system being relaxed, $\zeta = 1$, in the case under discussion, the generalized and effective anisotropy parameters, ζ_{pp} and $\widetilde{\zeta}_{pp}$, coincide with their counterparts related to the usual tensor virial equations, and Eqs. (46a) and (47a) take the explicit form (C06):

$$\zeta_{\ell\ell} = \frac{(1/3)T_{\rm rdm} - (2n/N)(n/2)m[(\overline{v_{\phi_r}})_n]^2}{T_{\rm rdm} - (2n/N)nm[(\overline{v_{\phi_r}})_n]^2} \quad , \qquad \ell = s, t \quad , \tag{65a}$$

$$\zeta_{rr} = \frac{(1/3)T_{\rm rdm}}{T_{\rm rdm} - (2n/N)nm[(\overline{v_{\phi_r}})_n]^2} \quad , \tag{65b}$$

where the special case, $n = 0$, relates to the initial configuration, characterized by isotropic random velocity component distributions ($\zeta_{pp} = 1/3$) and no figure rotation.

In the extreme case where the reversion process is completed, $n = N/2$, the changes expressed by Eqs. (59)-(64) take the form:

$$T_{\rm rdm} \to T_{\rm rdm} - \frac{N}{2}m[(\overline{v_{\phi_r}})_{N/2}]^2 \quad , \tag{66}$$

$$(T_{\rm rdm})_{\phi_r\phi_r} \to (T_{\rm rdm})_{\phi_r\phi_r} - \frac{N}{2}m[(\overline{v_{\phi_r}})_{N/2}]^2 \quad , \tag{67}$$

$$(T_{\rm rdm})_{\ell\ell} \to (T_{\rm rdm})_{\ell\ell} - \frac{N}{4}m[(\overline{v_{\phi_r}})_{N/2}]^2 \quad , \qquad \ell = s, t \quad ; \tag{68}$$

$$T_{\rm sys} \to 0 + \frac{N}{2}m[(\overline{v_{\phi_r}})_{N/2}]^2 \quad , \tag{69}$$

$$(T_{\rm sys})_{\phi_r\phi_r} \to 0 + \frac{N}{2}m[(\overline{v_{\phi_r}})_{N/2}]^2 \quad , \tag{70}$$

$$(T_{\rm sys})_{\ell\ell} \to 0 + \frac{N}{4}m[(\overline{v_{\phi_r}})_{N/2}]^2 \quad , \qquad \ell = s, t \quad . \tag{71}$$

Similarly, Eqs. (65) take the form:

$$\zeta_{\ell\ell} = \frac{(1/3)T_{\rm rdm} - (N/4)m[(\overline{v_{\phi_r}})_{N/2}]^2}{T_{\rm rdm} - (N/2)m[(\overline{v_{\phi_r}})_{N/2}]^2} \quad , \qquad \ell = s, t \quad ; \tag{72a}$$

$$\zeta_{rr} = \frac{(1/3)T_{\rm rdm}}{T_{\rm rdm} - (N/2)m[(\overline{v_{\phi_r}})_{N/2}]^2} \quad ; \tag{72b}$$

in any case, the anisotropy excess, $\zeta_{\ell\ell} - \zeta_{rr} < 0$, is counterbalanced by figure rotation.

4.2. Axial velocity component reversion

Performing the reversion process of axial velocity components of given fraction of particles, n/N, implies the conversion of random (translation) motion kinetic energy into systematic (translation) motion kinetic energy, as:

$$\Delta(T_{\rm rdm})_{rr} = -\Delta(T_{\rm sys})_{rr} = -\frac{1}{2}M(\overline{v_r})^2 = -\frac{2n}{N}nm[(\overline{v_r})_n]^2 \quad , \tag{73}$$

where the remaining parameters are left unchanged.

The reversion process implies the following energy changes:

$$T_{\rm rdm} \to T_{\rm rdm} - \frac{2n}{N}nm[(\overline{v_r})_n]^2 \quad , \tag{74}$$

$$(T_{\rm rdm})_{rr} \to (T_{\rm rdm})_{rr} - \frac{2n}{N}nm[(\overline{v_r})_n]^2 \quad , \tag{75}$$

$$T_{\rm sys} \to 0 + \frac{2n}{N}nm[(\overline{v_r})_n]^2 \quad , \tag{76}$$

$$(T_{\rm sys})_{rr} \to 0 + \frac{2n}{N}nm[(\overline{v_r})_n]^2 \quad , \tag{77}$$

while the contributions from random motions along the principal axes x_s and x_t, $(T_{\rm rdm})_{ss}$ and $(T_{\rm rdm})_{tt}$, remain unchanged.

With the system being relaxed, $\zeta = 1$, in the case under discussion, the generalized and effective anisotropy parameters, ζ_{pp} and $\widetilde{\zeta}_{pp}$, coincide with their counterparts related to the usual tensor virial equations, and Eqs. (46a) and (47a) take the explicit form:

$$\zeta_{\ell\ell} = \frac{(1/3)T_{\rm rdm}}{T_{\rm rdm} - (2n/N)nm[(\overline{v_r})_n]^2} \quad , \qquad \ell = s, t \quad ; \tag{78a}$$

$$\zeta_{rr} = \frac{(1/3)T_{\rm rdm} - (2n/N)nm[(\overline{v_r})_n]^2}{T_{\rm rdm} - (2n/N)nm[(\overline{v_r})_n]^2} \quad , \tag{78b}$$

where the special case, $n = 0$, refers to the initial configuration, characterized by isotropic random velocity component distributions ($\zeta_{pp} = 1/3$) and no figure rotation.

In the extreme case where the reversion process is completed, $n = N/2$, the changes expressed by Eqs. (74)-(77) take the form:

$$T_{\rm rdm} \to T_{\rm rdm} - \frac{N}{2}m[(\overline{v_r})_{N/2}]^2 \quad , \tag{79}$$

$$(T_{\rm rdm})_{rr} \to (T_{\rm rdm})_{rr} - \frac{N}{2}m[(\overline{v_r})_{N/2}]^2 \quad , \tag{80}$$

$$T_{\rm sys} \to 0 + \frac{N}{2}m[(\overline{v_r})_{N/2}]^2 \quad , \tag{81}$$

$$(T_{\rm sys})_{rr} \to 0 + \frac{N}{2}m[(\overline{v_r})_{N/2}]^2 \quad , \tag{82}$$

similarly, Eqs. (78) take the form:

$$\zeta_{\ell\ell} = \frac{(1/3)T_{\rm rdm}}{T_{\rm rdm} - (N/2)m[(\overline{v_r})_{N/2}]^2} \quad , \qquad \ell = s, t \quad ; \tag{83a}$$

$$\zeta_{rr} = \frac{(1/3)T_{\rm rdm} - (N/2)m[(\overline{v_r})_{N/2}]^2}{T_{\rm rdm} - (N/2)m[(\overline{v_r})_{N/2}]^2} \quad ; \tag{83b}$$

in any case, the anisotropy excess, $\zeta_{\ell\ell} - \zeta_{rr} > 0$, is counterbalanced by figure (imaginary) rotation.

4.3. Change of reference frame and imaginary rotation

Performing the reversion process on a given fraction of particles, n/N, implies the conversion of random motion (translation) kinetic energy into systematic motion (translation) kinetic energy, with respect to the principal axis x_r, according to Eqs. (73)-(77). The related kinetic-energy tensor component of the centre of inertia, by use of Eqs. (57) and (73), reads:

$$(T_{\rm C})_{rr} = \frac{1}{2}M(\overline{v_r})^2 = \frac{2n}{N}nm[(\overline{v_r})_n]^2 = -\Delta(T_{\rm rdm})_{rr}, \tag{84}$$

which, in the case under discussion, coincides with the kinetic energy of the centre of inertia, $T_{\rm C}$, since $(T_{\rm C})_{ss} = (T_{\rm C})_{tt} = 0$. In the centre of inertia reference frame, the random motion kinetic-energy tensor component related to the axis x_r, $(T'_{\rm rdm})_{rr}$, by use of Eqs. (57) and (84), after the reversion process takes the form:

$$(T'_{\rm rdm})_{rr} = \frac{1}{2}m\sum_{i=1}^{N}\left[v_r^{(i)} - v_{{\rm C}r}\right]^2 = (T_{\rm rdm})_{rr} - \frac{1}{2}M(\overline{v_r})^2 = (T_{\rm rdm})_{rr} - (T_{\rm C})_{rr}, \tag{85}$$

where the kinetic energy, $T_{\rm C} = (T_{\rm C})_{rr}$, is masked by the change of reference frame (e.g. LL66, Chap. II, § 8).

Let the i-th particle be at the distance, $w_r^{(i)} = \{[x_s^{(i)}]^2 + [x_t^{(i)}]^2\}^{1/2}$, from the principal axis x_r, with velocity component, $v_r^{(i)}$. The imaginary angular velocity (Caimmi 1996, C06, C07), $i\Omega_r^{(i)}$, can be defined in such a way that the translational kinetic energy along the axis x_r is counterbalanced by the imaginary rotational kinetic energy around the x_r axis, i.e.

$$\frac{1}{2}m[v_r^{(i)}]^2 + \frac{1}{2}m\left[w_r^{(i)}\right]^2\left[i\Omega_r^{(i)}\right]^2 = 0 \quad , \tag{86}$$

$$\Omega_r^{(i)} = \frac{v_r^{(i)}}{w_r^{(i)}} \quad , \tag{87}$$

where the index, i, refers to the i-th particle, and the factor, i, is the imaginary unit. In this context, the velocity components on the principal axis x_r may be translated into imaginary tangential velocity components in the $(\mathsf{O}\, x_s\, x_t)$ principal plane, as:

$$(iv_{\phi_r})^2 = (iw_r\Omega_r)^2 = -v_r^2 \quad ; \tag{88}$$

according to Eqs. (86) and (87).

Let imaginary rotation around the axis x_r be imparted to the particles where the reversion process has occurred, and their counterparts with equal imaginary tangential velocity components, as prescribed by Eq. (88), particularized to the mean axial velocity component, $\overline{v_r}$, expressed by Eq. (57). The related increment in imaginary rotational kinetic energy reads:

$$\Delta(T_{\rm sys})_{\phi_r\phi_r} = \frac{1}{2}M(\overline{iv_{\phi_r}})^2 = \frac{2n}{N}nm[(\overline{iv_{\phi_r}})_n]^2 \quad , \tag{89}$$

and the combination of Eqs. (73), (84), (88), and (89) yields:

$$\Delta(T_{\rm sys})_{\phi_r\phi_r} = -\Delta(T_{\rm sys})_{rr} = -(T_{\rm C})_{rr} = \Delta(T_{\rm rdm})_{rr}. \tag{90}$$

The above results may be reduced to a single statement.

Theorem 1. *Given an R fluid with isotropic random velocity distribution and no figure rotation, let the axial velocity component reversion process be performed on a given fraction of particles, n/N, with respect to the principal axis x_r. Then turning to the centre of inertia reference frame with a kinetic energy loss, $\Delta(T_{\rm sys})_{rr} = (T_{\rm C})_{rr}$, is equivalent to putting the initial configuration into imaginary rotation around the principal axis x_r, with square mean tangential velocity component, $(\overline{iv_{\phi_r}})^2 = -(\overline{v_r})^2$, implying a kinetic energy gain, $\Delta(T_{\rm sys})_{\phi_r\phi_r} = -(T_{\rm C})_{rr}$.*

Accordingly, Eqs. (74)-(77) are replaced by the following:

$$T_{\rm sys} \to 0 - \frac{2n}{N}nm[(\overline{v_{\phi_r}})_n]^2 \quad , \tag{91}$$

$$(T_{\rm sys})_{\phi_r\phi_r} \to 0 - \frac{2n}{N}nm[(\overline{v_{\phi_r}})_n]^2 \quad , \tag{92}$$

$$(T_{\rm sys})_{\ell\ell} \to 0 - \frac{1}{2}\frac{2n}{N}nm[(\overline{v_{\phi_r}})_n]^2 \quad , \tag{93}$$

while the contributions from random motions remain unchanged, and the random velocity distribution remains isotropic ($\zeta_{11} = \zeta_{22} = \zeta_{33} = 1/3$).

In the extreme case where the reversion process is complete, $n = N/2$, and the maximum amount of available imaginary rotation has been attained, the changes expressed by Eq. (91)-(93) take the form:

$$T_{\rm sys} \to 0 - \frac{N}{2}m[(\overline{v_{\phi_r}})_{N/2}]^2 \quad , \tag{94}$$

$$(T_{\rm sys})_{\phi_r\phi_r} \to 0 - \frac{N}{2}m[(\overline{v_{\phi_r}})_{N/2}]^2 \quad , \tag{95}$$

$$(T_{\rm sys})_{\ell\ell} \to 0 - \frac{N}{4}m[(\overline{v_{\phi_r}})_{N/2}]^2 \quad , \tag{96}$$

the above results may be reduced to a single statement.

Theorem 2. *Given an R fluid with isotropic random velocity distribution and no figure rotation, let the axial velocity component reversion process be performed on a given fraction of particles, n/N, with respect to the principal axis x_r, and the reference frame changed into the centre of inertia reference frame, then the resulting configuration with anisotropy excess, $\zeta_{\ell\ell} - \zeta_{rr} > 0$, Eqs. (78), is equivalent to the initial configuration with null anisotropy excess,*

$\zeta_{\ell\ell}-\zeta_{rr}=0$, and imaginary rotation around the principal axis x_r, with square mean tangential velocity component, $(\overline{iv_{\phi_r}})^2=-(\overline{v_r})^2$.

4.4. Anisotropy excess and imaginary rotation

Consider a nonrotating ($\widetilde{\Omega_r}=0$) isotropic ($\zeta_{11}=\zeta_{22}=\zeta_{33}=1/3$) configuration, and let the tangential velocity component on the principal plane $(\mathsf{O}\,x_s\,x_t)$ be reversed in such a way that Eqs. (66)-(72) hold. Let an equal amount of real and imaginary tangential velocity component on the principal plane $(\mathsf{O}\,x_s\,x_t)$ be imparted to each particle for a total contribution equal to $(2/3)f_rT_{\rm rdm}$ and $-(2/3)f_rT_{\rm rdm}$, respectively, where f_r is a positive real number, which leaves the total energy unchanged. Let the reversion process be repeated for real tangential velocity components, to attain a null figure (real) rotation. The related changes, with respect to the initial configuration, are:

$$T_{\rm rdm}\to\left(1+\frac{2}{3}f_r\right)T_{\rm rdm}\quad, \tag{97}$$

$$(T_{\rm rdm})_{\phi_r\phi_r}\to(T_{\rm rdm})_{\phi_r\phi_r}+\frac{2}{3}f_rT_{\rm rdm}\quad, \tag{98}$$

$$(T_{\rm rdm})_{\ell\ell}\to\frac{1}{2}\left(\frac{2}{3}+\frac{2}{3}f_r\right)T_{\rm rdm}\quad, \tag{99}$$

$$T_{\rm sys}\to 0-\frac{2}{3}f_rT_{\rm rdm}\quad, \tag{100}$$

$$(T_{\rm sys})_{\phi_r\phi_r}\to 0-\frac{2}{3}f_rT_{\rm rdm}\quad, \tag{101}$$

$$(T_{\rm sys})_{\ell\ell}\to 0-\frac{1}{2}\frac{2}{3}f_rT_{\rm rdm}\quad, \tag{102}$$

and the related anisotropy parameters read:

$$\zeta_{\ell\ell}=\frac{[(1/3)+(1/3)f_r]T_{\rm rdm}}{[1+(2/3)f_r]T_{\rm rdm}}=\frac{1+f_r}{3+2f_r}\quad,\qquad \ell=s,t\quad; \tag{103a}$$

$$\zeta_{rr}=\frac{(1/3)T_{\rm rdm}}{[1+(2/3)f_r]T_{\rm rdm}}=\frac{1}{3+2f_r}\quad, \tag{103b}$$

where the anisotropy excess, $\zeta_{\ell\ell}-\zeta_{rr}=f_r/(3+2f_r)>0$, is counterbalanced by imaginary rotation, according to an initial configuration with isotropic velocity component distribution and no figure rotation. As application of the above results, two significative examples will be considered.

First example. *Systems flattened by anisotropic velocity component distribution ($\sigma_{11}=\sigma_{22}>\sigma_{33}$) with no figure rotation.*

Let tangential velocity components on the principal plane $(\mathsf{O}\,x_1\,x_2)$ be reversed in a convenient fraction of particles, n/N, to yield a convenient amount of figure rotation together with isotropic velocity component distribution ($\sigma'_{11}=\sigma'_{22}=\sigma_{33}$), as sketched in Fig. 1.

The combination of Eqs. (53), (54), and (56a) yields:

$$(\widetilde{\Omega_r})^2=\frac{1}{R_{Gr}^2}\frac{4n^2}{N^2}[(\overline{v_{\phi_r}})_n]^2\quad; \tag{104}$$

$$\left(\sigma_{\widetilde{\Omega_r}\widetilde{\Omega_r}}\right)^2=\frac{1}{R_{Gr}^2}\left(\sigma_{\phi_r\phi_r}\right)^2=\frac{1}{R_{Gr}^2}\left\{\overline{(v_{\phi_r}^2)}-\frac{4n^2}{N^2}[(\overline{v_{\phi_r}})_n]^2\right\}\quad; \tag{105}$$

while the mean square velocity components are left unchanged in the reversion process. The substitution of Eqs. (104) and (105) into (38b) and (38c) shows the dependence of systematic and random motion tangential kinetic-energy tensor components on the reversion process.

In the case under discussion ($r=3$), the random velocity component distribution has to be isotropic after the reversion process, which makes Eqs. (61) and (64) to reduce to:

$$(T_{\rm rdm})_{\ell\ell}=\frac{M}{2}\left[\sigma_{\ell\ell}^2-\left(\sigma_{\ell\ell}^2-\sigma_{33}^2\right)\right]=\frac{M}{2}\sigma_{33}^2\quad;\qquad \ell=1,2\quad; \tag{106}$$

$$(T_{\rm sys})_{\ell\ell}=\frac{M}{2}\left(\sigma_{\ell\ell}^2-\sigma_{33}^2\right)\quad;\qquad \ell=1,2\quad; \tag{107}$$

$$\frac{2n}{N}nm[\overline{(v_{\phi_r})_n}]^2=\frac{1}{2}\frac{4n^2}{N^2}M[(\overline{v_{\phi_r}})_n]^2=\frac{M}{2}(\overline{v_{\phi_r}})^2=\frac{M}{2}\left(\sigma_{\ell\ell}^2-\sigma_{33}^2\right); \tag{108}$$

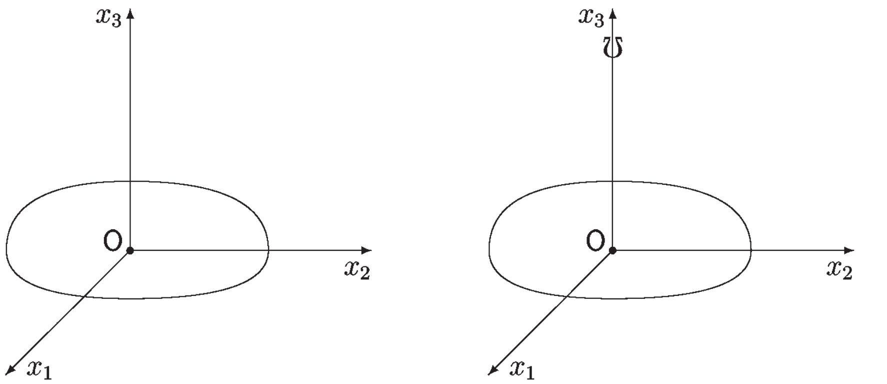
which defines the configuration after the reversion process.

Second example. *Systems elongated by anisotropic velocity component distribution ($\sigma_{11}=\sigma_{22}<\sigma_{33}$) with no figure rotation.*

Let a convenient real and imaginary figure rotations, $\overline{v_{\phi_3}}$ and $\overline{iv_{\phi_3}}$, with respect to the principal axis x_3, be imparted to the system. Then the kinetic energy remains unchanged. Concerning the real rotation, let the reversion process be performed on one half of the particles, leaving figure imaginary rotation together with isotropic velocity component distribution ($\sigma'_{11}=\sigma'_{22}=\sigma_{33}$), as sketched in Fig. 2.

Accordingly, Eqs. (104) and (105) hold for imaginary tangential velocity components on the principal plane $(\mathsf{O}\,x_1\,x_2)$ and figure imaginary rotation around the principal axis x_3.

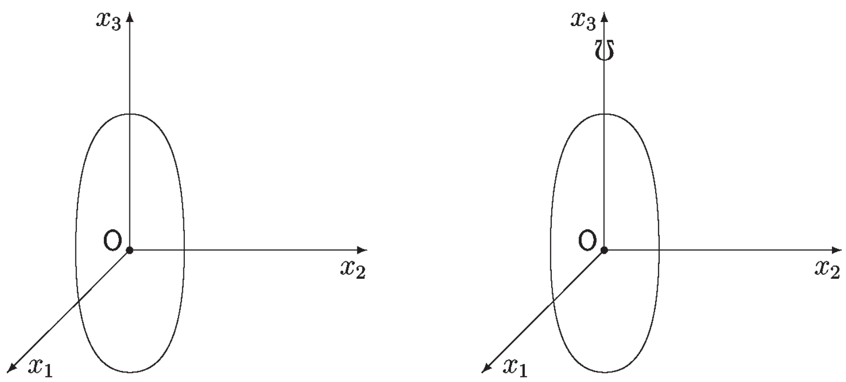
In the case under discussion ($r=3$), an isotropic velocity component distribution after the reversion process implies the validity of Eqs. (106), (107), and (108), where the tangential velocity components are imaginary ($\sigma_{\ell\ell}<\sigma_{33}$), and the configuration after the reversion process is completely defined.



$\sigma_{11} = \sigma_{22} > \sigma_{33}$; $\sigma'_{11} = \sigma'_{22} = \sigma_{33}$;

$(\overline{v_{\phi_3}})^2 = 0$; $(\overline{v_{\phi_3}})^2 = \sigma_{11}^2 - \sigma_{33}^2 = \sigma_{22}^2 - \sigma_{33}^2$;

Fig. 1. *After reversion of tangential velocity components for a convenient fraction of particles, with respect to the principal plane ($\mathsf{O}x_1x_2$), a configuration flattened by anisotropic velocity component distribution ($\sigma_{11} = \sigma_{22} > \sigma_{33}$) with no figure rotation ($\overline{v_{\phi_3}} = 0$), (left), is turned into a configuration flattened by figure rotation $\overline{v_{\phi_3}} = (\sigma_{11}^2 - \sigma_{33}^2)^{1/2} = (\sigma_{22}^2 - \sigma_{33}^2)^{1/2}$ with isotropic velocity component distribution ($\sigma'_{11} = \sigma'_{22} = \sigma_{33}$), (right). The symbol, ℧, denotes figure rotation about the corresponding principal axis.*



$\sigma_{11} = \sigma_{22} < \sigma_{33}$; $\sigma'_{11} = \sigma'_{22} = \sigma_{33}$;

$(\overline{\mathrm{i}v_{\phi_3}})^2 = 0$; $(\overline{\mathrm{i}v_{\phi_3}})^2 = \sigma_{11}^2 - \sigma_{33}^2 = \sigma_{22}^2 - \sigma_{33}^2$;

Fig. 2. *After imparting a convenient figure real and imaginary rotations, $\overline{v_{\phi_3}}$ and $\overline{iv_{\phi_3}}$, with respect to the principal axis x_3, to the system, and reversing real tangential velocity components on one half of the particles, a configuration elongated by anisotropic velocity component distribution ($\sigma_{11} = \sigma_{22} < \sigma_{33}$) with no figure rotation ($\overline{v_{\phi_3}} = 0$), (left), is turned into a configuration elongated by figure (imaginary) rotation $\overline{iv_{\phi_3}} = (\sigma_{11}^2 - \sigma_{33}^2)^{1/2} = (\sigma_{22}^2 - \sigma_{33}^2)^{1/2}$ with isotropic velocity component distribution ($\sigma'_{11} = \sigma'_{22} = \sigma_{33}$), (right). The symbol, ℧, denotes figure rotation about the corresponding principal axis.*

4.5. Tangential velocity component reversion in the general case

In the general case of anisotropic velocity component distribution ($\sigma_{11} \neq \sigma_{22} \neq \sigma_{33}$) and figure rotation around each principal axis ($\widetilde{\Omega_1} \neq \widetilde{\Omega_2} \neq \widetilde{\Omega_3} \neq 0$), with regard to the principal plane ($\mathsf{O}\, x_s\, x_t$), let the tangential velocity component reversion process be applied to one half of the particles in such a way that no figure rotation about the principal axis x_r occurs, $\widetilde{\Omega_r} = 0$. Keeping in mind Eqs. (51)-(54), the corresponding energy changes read:

$$(T_{\rm rdm})_{\phi_r\phi_r} \to (T_{\rm rdm})_{\phi_r\phi_r} + \frac{1}{2}M(\overline{v_{\phi_r}})^2 \quad , \quad (109)$$

$$(T_{\rm sys})_{\phi_r\phi_r} \to (T_{\rm sys})_{\phi_r\phi_r} - \frac{1}{2}M(\overline{v_{\phi_r}})^2 = 0 \quad , (110)$$

$$\overline{v_{\phi_r}} = (\overline{v_{\phi_r}})_{N/2} = \frac{1}{N}\sum_{i=1}^{N} v_{\phi_r}^{(i)} \quad , \quad (111)$$

$$\Delta(T_{\rm sys})_{\phi_r\phi_r} = -\Delta(T_{\rm rdm})_{\phi_r\phi_r} = -\frac{1}{2}M(\overline{v_{\phi_r}})^2 = -\frac{1}{2}MR_{Gr}^2(\widetilde{\Omega_r})^2 \quad , \quad (112)$$

$$[(T_{\rm rdm})_{\phi_r\phi_r}]_{\ell\ell} + [(T_{\rm rdm})_{w_rw_r}]_{\ell\ell} = \frac{1}{2}M\sigma_{\ell\ell}^2 \quad , \quad \ell = 1,2 \quad ; \quad (113)$$

$$(T_{\rm rdm})_{rr} = \frac{1}{2}M\sigma_{rr}^2 \quad , \quad (114)$$

where the rms velocity components, $\sigma_{\ell\ell}^2$ and σ_{rr}^2, pertain to the initial configuration.

The changes in anisotropy parameters, $\zeta_{pp} = \sigma_{pp}^2/\sigma^2$, read:

$$\zeta_{\ell\ell} \to \frac{\zeta_{\ell\ell} + (1/2)\Delta\zeta_{\ell\ell}}{1 + \Delta\zeta_{\ell\ell}} \quad ; \qquad \ell = s,t \quad ; \quad (115a)$$

$$\zeta_{rr} \to \frac{\zeta_{rr}}{1 + \Delta\zeta_{\ell\ell}} \quad ; \quad (115b)$$

$$\Delta\zeta_{\ell\ell} = \frac{(\overline{v_{\phi_r}})^2}{\sigma^2} \quad ; \qquad \ell = s,t \quad ; \quad (115c)$$

where the rms velocity, σ^2, pertain to the initial configuration.

The application of the above procedure to the principal axes x_1 and x_2, makes the transition from an initial configuration with rms velocity components σ_{11}^2, σ_{22}^2, σ_{33}^2, and figure rotation around the principal axes $\widetilde{\Omega_1}$, $\widetilde{\Omega_2}$, $\widetilde{\Omega_3}$, to a final configuration with rms velocity components $(\sigma_{11}')^2$, $(\sigma_{22}')^2$, $(\sigma_{33}')^2$, and figure rotation around the principal axes 0, 0, $\widetilde{\Omega_3}$. The corresponding energy changes read:

$$(T_{\rm rdm})_{11} \to (T_{\rm rdm})_{11} + \frac{1}{4}M(\overline{v_{\phi_2}})^2 \quad , \quad (116)$$

$$(T_{\rm rdm})_{22} \to (T_{\rm rdm})_{22} + \frac{1}{4}M(\overline{v_{\phi_1}})^2 \quad , \quad (117)$$

$$(T_{\rm rdm})_{33} \to (T_{\rm rdm})_{33} + \frac{1}{4}M\left[(\overline{v_{\phi_1}})^2 + (\overline{v_{\phi_2}})^2\right] , \quad (118)$$

$$T_{\rm rdm} \to T_{\rm rdm} + \frac{1}{2}M\left[(\overline{v_{\phi_1}})^2 + (\overline{v_{\phi_2}})^2\right] \quad , \quad (119)$$

$$(T_{\rm rdm})_{pp} = \frac{1}{2}M\sigma_{pp}^2 \quad , \qquad p = 1,2,3 \quad , \quad (120)$$

and the pertaining changes in anisotropy parameters are:

$$\zeta_{11} \to \frac{\zeta_{11} + (1/2)\Delta\zeta_{11}}{1 + \Delta\zeta_{11} + \Delta\zeta_{22}} \quad , \quad (121a)$$

$$\zeta_{22} \to \frac{\zeta_{22} + (1/2)\Delta\zeta_{22}}{1 + \Delta\zeta_{11} + \Delta\zeta_{22}} \quad , \quad (121b)$$

$$\zeta_{33} \to \frac{\zeta_{33} + (1/2)[\Delta\zeta_{11} + \Delta\zeta_{22}]}{1 + \Delta\zeta_{11} + \Delta\zeta_{22}} \quad , \quad (121c)$$

$$\Delta\zeta_{11} = \frac{(\overline{v_{\phi_2}})^2}{\sigma^2} \quad , \qquad \Delta\zeta_{22} = \frac{(\overline{v_{\phi_1}})^2}{\sigma^2} \quad . (121d)$$

The above results may be reduced to a single statement.

Theorem 3. *Given an R fluid, a convenient application of the tangential velocity component reversion process produces an adjoint configuration where figure rotation occurs around a single principal axis, that is an R3 fluid.*

Accordingly, the results valid for R3 fluids (C07) may be extended to the general case of R fluids.

5. DISCUSSION

As suggested in earlier attempts (Caimmi 1996, C06, C07), the equivalence between a variation in figure rotation and in anisotropy excess, may provide a useful tool for the description of collisionless fluids. The discussion here shall be focused on the spin parameter (Peebles 1969, 1971):

$$\lambda^2 = -\frac{J^2E}{G^2M^5} \quad ; \quad (122)$$

where G is the gravitational constant, M the total mass, J the total angular momentum, and E the total energy. The above formulation includes four possibilities, namely (i) real rotation ($J^2 \geq 0$) and bound system ($E < 0$), which is the sole currently used in literature; (ii) imaginary rotation ($J^2 < 0$) and bound system ($E < 0$); (iii)real rotation ($J^2 \geq 0$) and unbound system ($E \geq 0$); (iv) imaginary rotation ($J^2 < 0$) and unbound system ($E \geq 0$). Accordingly, the spin parameter attains real values in cases (i) and (iv), and imaginary values in cases (ii) and (iii).

In the light of the current model, oblate-like and prolate-like configurations belong to cases (i) and (ii) outlined above, while cases (iii) and (iv) represent unbound structures for which the virial equations do not hold. Therefore, the comparison with observations and/or computations must be restricted to bound configurations.

The spin parameter distribution is usually fitted using a lognormal distribution (e.g. van den Bosh 1998, Gardner 2001, Ballin and Steinmetz 2005, Hernandez et al. 2007) or, in general, one dependent on $\log\lambda$ (e.g. Bett et al. 2007), in dealing with real rotation. The inclusion of imaginary rotation would imply use of λ^2 instead of λ as independent variable, allowing for both positive (real rotation) and negative (imaginary rotation) values.

The following procedure should be followed for calculating λ^2 from observations and/or computations: (1) determine the inertia tensor and the principal axes of inertia for a given matter distribution; (2) determine the potential-energy tensor; (3) determine the anisotropy parameters using the generalized virial equations; (4) perform the reversion process with respect to two principal axes of inertia to leave figure rotation around the third one (R3 fluid); (5) convert the anisotropy excess into figure (real or imaginary) rotation to obtain isotropic velocity component distribution ($\zeta_{11} = \zeta_{22} = \zeta_{33}$); (6) evaluate the spin parameter; (7) act as already done for all the sample objects; (8) determine the distribution of the spin parameter, $P(\lambda^2)$, with respect to the sample of adjoint configurations, where the velocity component distribution is isotropic.

The related results could provide further insight on formation and evolution of large-scale collisionless fluids, such as galaxies and galaxy clusters.

6. CONCLUSION

A theory of collisionless fluids was developed in a unified picture, where the nonrotating ($\widetilde{\Omega_1} = \widetilde{\Omega_2} = \widetilde{\Omega_3} = 0$) figures with specified random velocity component distributions, and the rotating ($\widetilde{\Omega_1} \neq \widetilde{\Omega_2} \neq \widetilde{\Omega_3}$) ones with different random velocity component distributions, make adjoint configurations to the same system. R fluids have been defined as ideal, self-gravitating fluids satisfying the virial theorem assumptions (e.g. LL66, Chap. II, § 10; C07), in presence of figure rotation around each principal axis of inertia.

To this aim, mean and rms angular velocities and mean and rms tangential velocity components have been expressed, by weighting on the moment of inertia and the mass, respectively. The figure rotation has been defined as the mean angular velocity, weighted on the moment of inertia, with respect to a selected axis.

The generalized tensor virial equations (Caimmi and Marmo 2005) were formulated for R fluids and further attention was devoted to axisymmetric configurations where, for selected coordinate axes, a variation in figure rotation has to be counterbalanced by a variation in anisotropy excess and vice versa.

A microscopical analysis of systematic and random motions was performed under a number of general hypotheses, by reversing the sign of tangential or axial velocity components of an specified fraction of particles, leaving the distribution function and other parameters unchanged (Meza 2002).

The application of the reversion process to tangential velocity components, was found to imply the conversion of random motion rotation kinetic energy into that of systematic motion. The application of the reversion process to axial velocity components was found to imply the conversion of random motion translation kinetic energy into systematic motion translation kinetic energy, and the loss related to a change of reference frame has been expressed in terms of systematic motion (imaginary) rotation kinetic energy.

A number of special situations was investigated in more detail. It was found that an R fluid always allows an adjoint configuration where figure rotation takes place about only one principal axis of inertia (R3 fluid), which implies that all the results related to R3 fluids (Caimmi 2007) may be extended to R fluids.

Finally, a procedure has been sketched for deriving the spin parameter distribution (including imaginary rotation) from a sample of observed or simulated large-scale collisionless fluids i.e. galaxies and galaxy clusters.

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Развијена је обједињена теорија безсударног флуида у којој неротирајућа тела ($\widetilde{\Omega_1} = \widetilde{\Omega_2} = \widetilde{\Omega_3} = 0$) са приписаном расподелом случајних брзина компонената и ротирајућа тела ($\widetilde{\Omega_1} \neq \widetilde{\Omega_2} \neq \widetilde{\Omega_3}$) са различитом расподелом случајних брзина компонената чине адјунговане конфигурације истог система. R флуиди су дефинисани као идеални, самогравитирајући флуиди који задовољавају претпоставке теореме виријала, са систематском ротацијом око сваке од главних оса инерције. Средња угаона брзина и стандардна девијација, као и средња тангенцијална брзина и стандардна девијација сваке од компонената изражене су тако да су момент инерције и маса представљене одговарајућим тежинама. Ротација тела је дефинисана преко средње угаоне брзине са отежињеним моментом инерције у односу на одабрану осу. Постављене су генерализоване тензорске виријалне једначине (Caimmi and Marmo 2005) за R флуиде и даља пажња је посвећена осносиметричним конфигурацијама код којих, за изабране координатне осе, варијације у ротацији тела морају бити уравнотежене варијацијама у мери анизотропије, и обрнуто. Микроскопска анализа систематског и неуређеног кретања извршена је у складу са одређеним општим хипотезама, изменом знака тангенцијалних и аксијалних компонената брзине изабраног скупа честица, остављајући при том функцију расподеле и друге параметре непромењеним (Meza 2002). Из примене процедуре измене знака тангенцијалних компонената брзина следи конверзија кинетичке енергије неуређеног кретања у кинетичку енергију систематског ротационог кретања. Из примене процедуре измене знака аксијалних компонената брзина следи конверзија кинетичке енергије транслације неуређеног кретања у кинетичку енергију транслације систематског кретања, док је губитак повезан са променом система референце представљен преко кинетичке енергије (имагинарног) систематског ротационог кретања. Детаљно је проучен и одређен број специјалних случајева. Нађено је да R флуид увек дозвољава адјунговану конфигурацију код које се ротација тела одвија око само једне главне осе инерције (R3 флуид), одакле следи да се сви резултати добијени за R3 флуиде (Caimmi 2007) могу проширити на R флуиде. На крају, скицирана је процедура за извођење функције расподеле параметра спина (укључујући имагинарну ротацију) на основу узорка посматраних или симулираних безсударних флуида на великим скалама, тј. галаксија и јата галаксија.