RECONSIDERATION OF MASS-DISTRIBUTION MODELS

S. Ninković

Astronomical Observatory, Volgina 7, 11060 Belgrade 38, Serbia
E–mail: *sninkovic@aob.rs*

SUMMARY: The mass-distribution model proposed by Kuzmin and Veltmann (1973) is revisited. It is subdivided into two models which have a common case. Only one of them is subject of the present study. The study is focussed on the relation between the density ratio (the central one to that corresponding to the core radius) and the total-mass fraction within the core radius. The latter one is an increasing function of the former one, but it cannot exceed one quarter, which takes place when the density ratio tends to infinity. Therefore, the model is extended by representing the density as a sum of two components. The extension results into possibility of having a correspondence between the infinite density ratio and 100% total-mass fraction. The number of parameters in the extended model exceeds that of the original model. Due to this, in the extended model, the correspondence between the density ratio and total-mass fraction is no longer one-to-one; several values of the total-mass fraction can correspond to the same value for the density ratio. In this way, the extended model could explain the contingency of having two, or more, groups of real stellar systems (subsystems) in the diagram total-mass fraction versus density ratio.

Key words. galaxies: structure, galaxies: kinematics and dynamics

1. INTRODUCTION

The steady state and spherical symmetry are often assumed as the first approximation for a number of stellar systems. A number of models, i.e. particular cases, satisfying these conditions have been proposed. Among them is the isochrone model. Historically, the isochrone model played an important role in studying various stellar systems. This is largely due to a property of its (see below), indicated for the first time by M. Hénon. More details about this property can be found, e.g. in Binney and Tremaine (1987, p. 109). Besides, the isochrone potential was assumed by Eggen et al. (1962) in their well-known study concerning the evolution of the Milky Way.

There exists a generalisation of the isochrone model (Kuzmin and Veltmann 1973, pp. 305-306) usually referred to as the generalised isochrone model (GIM). GIM has two parameters with the dimension of length. Therefore, there are two extremal cases corresponding to the smallest possible values of the parameters - zero. One of them had been known before Kuzmin and Veltmann published their paper. It is known as Schuster's or Plummer's mass distribution and it is an exception, because in the periphery the exponent of the power law followed by the density decrease is equal to 5. Otherwise, for GIM this exponent equals to 4. The other extremal case was rediscovered by Hernquist (1990). It is a special case of a family proposed independently by Dehnen (1993) and by Tremaine et al. (1994). It is known to yield a very simple expression for the potential which

has been used many times by the authors of models describing the mass distributions in the Milky Way and similar stellar systems, mainly to describe the bulge contribution (e.g. Law and Majewski 2010).

In the present paper, GIM is replaced with two models for which the classical isochrone model appears as the common case and each of them contains only one of the two extremal cases (clearly, not the same).

The present author has already studied one of the two submodels to find a particular case which is very similar to the mass distribution obtainable from the Schuster (or Plummer) density law after changing the exponent (Ninković 2001).

The present paper is devoted to the other submodel of GIM. The reason is its property that the increase of the total-mass fraction within the core radius is due to steepening of the density profile within the core. The submodel is extended in a way to enable: i) that the total-mass fraction tends to unity when the density ratio - central density to that at the core radius - tends to infinity; ii) that, due to a larger number of parameters, the relation between the total-mass fraction within the core and the density ratio (central to that at the core radius) is no longer unique.

The paper is organised in the following way. In Section 2, entitled Theoretical Base, the problem is posed in more detail. The results of the present study are in given in Section 3, entitled Procedure and Results. The results are discussed in Section 4, entitled Discussion. Finally, the basic conclusions are the subject of Section 5, entitled Conclusion.

2. THEORETICAL BASE

In the case of GIM the formula for potential is more simple than that describing the density. Its form is:

$$\Pi(r) = \frac{G\mathcal{M}}{r_1 + \sqrt{r_2^2 + r^2}} \ . \qquad (1)$$

Here, G is the universal gravitational constant, $\mathcal{M}$ is the total mass, r_1 and r_2 are the two scales. For the classical isochrone model they are equal. In the generalisation, it is allowed that these two quantities differ. The two extremal cases arise when either of them is equal to zero: $r_1 = 0$, the Schuster (Plummer) mass distribution, $r_2 = 0$, the limiting case (Kuzmin and Veltmann 1973), rediscovered by Hernquist (1990). Eq. (1) is written here in the same way as in an earlier paper of the present author (Ninković 1998, p. 19). The only difference is in the designations of the scales. This way of writing it, as a more clear one, is preferred to that used by the proposers (Kuzmin and Veltmann 1973, p. 301, Eq. (5.1) and p. 304, Eq. (5.18)).

The two (sub)models mentioned above would be: $r_1 \leq r_2$ and vice versa. Each of the two extremal cases belongs to only one of these models, the correspondence is obvious.

On the basis of Eq. (1) the corresponding formulae for the cumulative mass ($\mathcal{M}_r$) and density (ρ)

will be:

$$\mathcal{M}_r = \mathcal{M}\frac{r^3}{(r_1 + \sqrt{r_2^2 + r^2})^2 \sqrt{r_2^2 + r^2}} \ . \qquad (2)$$

$$\rho = \frac{\mathcal{M}}{4\pi}\frac{3r_1 r_2^2 + 2r_1 r^2 + 3r_2^2\sqrt{r_2^2 + r^2}}{(r_1 + \sqrt{r_2^2 + r^2})^3 (r_2^2 + r^2)^{3/2}} \ . \qquad (3)$$

Using these formulae one can easily obtain, if $r_1 = r_2$, that then within r_1, about 12% of the total mass is contained and that the ratio of the density at the centre to that at r_1 is about 3.23. If the case $r_1 \geq r_2$ is examined, then as the ratio r_1/r_2 increases, both the total-mass fraction within the larger scale (r_1) and the ratio of the densities corresponding to $r = 0$ and $r = r_1$ will also increase. In the extremal case $r_2 = 0$, the total-mass fraction attains exactly $1/4$, but the density ratio becomes infinite because the central density tends to infinity. In the other case, where $r_2 \geq r_1$, also with increasing ratio r_2/r_1 both the total-mass fraction and the density ratio (this time within and at exactly $r = r_2$, because it is more important) increase, but neither attains infinite values. In particular, for $r_1 = 0$ the total-mass fraction within r_2 is about 35% and the density at the centre is about 5.66 times that at $r = r_2$. These properties can be more clearly seen in Fig. 2. Since it is clear that in the extremal cases r_1 and r_2 must act as the core radii, it can be seen that for the model $r_1 \leq r_2$ the density within the core ($r = r_2$) is almost constant, whereas, nevertheless, the total-mass fraction within it increases rather sharply, even more sharply than in the alternative case. Therefore, one could conclude that in the case $r_1 \geq r_2$, the increase of the total-mass fraction within the core is due to the steepening of the density profile within the core, whereas the case $r_1 \leq r_2$ is characterised by a severe density decrease beyond the core. It is easily seen from Eq. (3) that the density derivative ($\frac{d\rho}{dr}$) equals zero at the centre. Though this may be interpreted as a core with constant density, based on what was already said above, it is clear that for $r_1 \geq r_2$ the density within the core is far from constant. The following example, in which $r_2/r_1 = 0.5$, yields that the density at r_1 is 9.87 times smaller than at the centre, whereas, at $2r_1$, it is 6.6 times smaller than at r_1.

In the reverse case, $r_2/r_1 = 2$, the density at $r = r_2$ is 3.91 times smaller than at the centre and at $r = 2r_2$ it is 6.41 times smaller than at $r = r_2$.

Two dimensionless quantities characterising the model: the total-mass fraction within the core radius and the density ratio - central density divided by that at the core radius - may be related. For both there are limits. Since the density is expected to be a decreasing function of the distance from the centre, the lower limit for the density ratio is 1. On the other hand, it can be arbitrarily large; as a consequence the upper limit is infinity. The total-mass fraction must be between 0 and 1. More precisely, the intervals of possible values for the density ratio and total-mass fraction are $(1, \infty)$ and $(0, 1)$, respectively.

The model $r_1 \geq r_2$ viewed in the light of this relation results into the following. The lower limits, corresponding to $r_1 = r_2$, for the mass fraction within r_1 and density ratio $\rho(0)/\rho(r_1)$ (0.121 and 3.23, respectively) agree fairly well with the intervals of possible values. As the ratio r_2/r_1 decreases, both dimensionless quantities increase. When the ratio r_2/r_1 tends to zero, the density ratio tends to infinity, in accordance with the interval of possible values, but the total-mass fraction within r_1 attains the value of 1/4 only. The plot of the total-mass fraction versus density ratio would have a horizontal asymptote at this value. This property can be seen in Fig. 2. Besides, since both quantities depend on the same quantity i.e. the ratio r_2/r_1 ($r_2/r_1 \leq 1$), there exists only one particular relation between the total-mass fraction within the core and the density ratio. In other words, to every value of the density ratio only one value of the total-mass fraction corresponds.

In the diagram of the total-mass fraction within the core radius versus density ratio, a real stellar system (subsystem) would be represented as a single point. There is no reason why many such points would follow a strictly established trend. In other words, one may expect several values of the total-mass fraction to correspond to the same value of the density ratio. We could have two, or more, groups of stellar systems where each of them would follow a different kind of dependence of the total-mass fraction within the core radius versus density ratio.

Therefore, it is desirable to extend the model $r_1 \geq r_2$ (Eqs. (1)-(3)). The extended model should make it possible to i) push the horizontal asymptote towards unity, which is in accordance with the intervals of possible values; ii) avoid existence of only one particular relation between the density ratio and total-mass fraction.

3. PROCEDURE AND RESULTS

A system (subsystem) in which the mass distribution obeys the model $r_1 \geq r_2$ (Eqs. (1)-(3)) is replaced by a two-component system. The components will be referred to as the inner one and outer one. They have a common centre. The steady state and spherical symmetry are valid for both. The only difference is in the mass distribution. The outer component follows the isochrone potential, more precisely Eqs. (1) to (3) with $r_1 = r_2$. The volume occupied by its mass is infinite. The inner component occupies a finite volume, inside a sphere. Its radius may be equal to the core radius of the outer component, r_1 (for the outer component r_1 and r_2 are equal).

Let the core radius r_1 for the outer component be the core radius for the system as a whole. Clearly, the total-mass fraction within it, φ, is then given by the following expression

$$\varphi = \frac{\kappa' + \nu}{\kappa + 1} \ . \tag{4}$$

Here κ is the total-mass ratio (inner-to-outer) for the two components, κ' is the mass of the inner component, for which the unit is the total mass of the outer component, contained within r_1, obviously there must be $\kappa' \leq \kappa$, and, finally, ν is the mass of the outer component, also with the total mass of this component as the unit, contained within r_1. Since the isochrone model ($r_1 = r_2$) is assumed for the outer component, the quantity ν becomes constant. Using Eq. (2) one easily obtains ν=0.121. The extremal case φ tends to 1, according to Eq. (4), is achieved when κ' approaches κ and then both tend to infinity.

Now the mass distribution for the inner component can be given in more detail. Its role is to contribute to the enlarging of the ratio - central density to that at the core radius (r_1), but provided that this enlarging is followed by the corresponding enlarging of φ, the fraction of the total mass of the system within the core. Quantitatively the significance of the inner component is expressed through the ratio κ. The initial values (κ =0), as already said, are $\rho(0)/\rho(r_1)$ =3.23 and φ =0.121. One should find such a mass distribution for which when κ tends to infinity the density ratio also becomes infinite and φ tends to 1 (Eq. (4)). In other words, in the case of the inner component the central density and the total mass become infinite simultaneously.

Such requirements can be satisfied by using the mass distribution with density given by the following expression:

$$\rho = \mathcal{M}_0 \left[\frac{1}{(r_i^2 + r^2)^{3/2}} - \frac{1}{(r_i^2 + r_e^2)^{3/2}} \right] \ . \tag{5}$$

This distribution should be used within a finite radius, which has been already assumed for the inner component. The scale r_i compared to r_1 can be arbitrarily small. In the limiting case, it tends to zero and then both, the central density $\rho(0)$ and the total mass of the inner component, become infinite. The outer radius of the inner component, here denoted by r_e, as said earlier, should be $\geq r_1$. The constant negative term containing r_e is added here to avoid the density discontinuity at the outer radius of the inner component. The distance to the system centre, at which the density described in this way attains zero, is the outer radius of the inner component. All relevant details concerning this mass distribution (cumulative mass, etc) can be found in Ninković (1998).

Thus, the mass distribution within the inner component is described by three parameters: the constant $\mathcal{M}_0$ having the dimension of mass and can be replaced by the total mass, the scale r_i and the outer radius r_e. If added the total mass of the outer component and the radius r_1, this would yield five parameters for the system as a whole. However, all these parameters are not completely independent; the outer radius of the inner component cannot be smaller than r_1, whereas r_i is required to be smaller than r_1.

The role of the inner component is twofold. Firstly, to enlarge the ratio of the central density to

that at r_1. In Fig. 1 two curves are presented. One of them corresponds to the case without the inner component. It can be seen from this figure how the presence of the inner component can arbitrarily enlarge the density ratio. Secondly, due to the inner component, the total-mass fraction within r_1 can attain various values tending to unity as the highest possible value. The total number of parameters, five, regardless to the limitations mentioned in the preceding paragraph, offers the possibility of obtaining at least two values for the total-mass fraction which correspond to the same value for the argument, density ratio, as presented in Fig. 2. Any curve of this kind has the same horizontal asymptote because the extremal values correspond to each other; when the density ratio tends to infinity, then the total-mass fraction tends to unity. For the purpose of comparison, the third curve given in Fig. 2 corresponds to the model defined by Eqs. (1)-(3), the case $r_2 \leq r_1$. As said earlier, in that case there exists only one particular relation between the total-mass fraction and the density ratio, determined by the ratio r_2/r_1 for which the horizontal asymptote is at the value of 0.25.
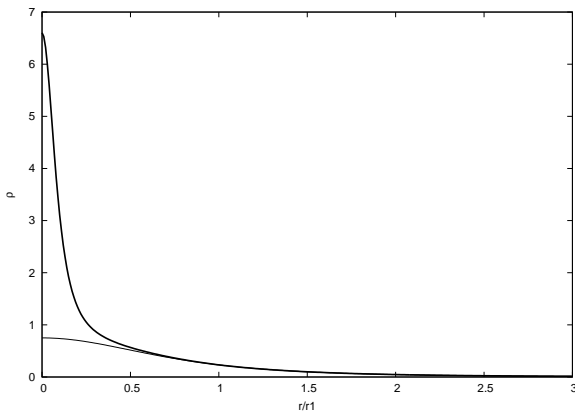

Fig. 1. *Thin curve: the density dependence on distance for the case of Eq. (3), $r_2 = r_1$, no inner component; thick curve: with inner component. Units are $\mathcal{M}/(4\pi r_1^3)$, $\mathcal{M}$ total mass of the outer component, for density, $\mathcal{M}/(4\pi)$ for mass, r_1 for distance; values of inner-component parameters: $\mathcal{M}_0 = 0.0078$, $r_i = 0.11$, $r_e = 1$.*

4. DISCUSSION

The relation between the density ratio (central to that at the core radius) and the total-mass fraction within the core is meaningful to be considered only if the density dependence on r is defined within an infinite radius. If the outer radius is finite, then the ratio r_o/r_c (where r_o is the outer radius) affects the fraction of the total mass within the core radius (r_c).

Density dependences valid for an infinite volume in the very outer parts can be usually approximated by means of simple laws, say a power law or exponential law, where in the case of a power

law the exponent modulus must exceed the value of three. In this way, the improper integral yielding the total mass will not result in an infinite value. The value of four for the exponent modulus, as sufficiently suitable, has charaterised many density dependences which are applied in infinite volume. As said above, this is exactly the case with the outer component in the model analysed in Section 3.
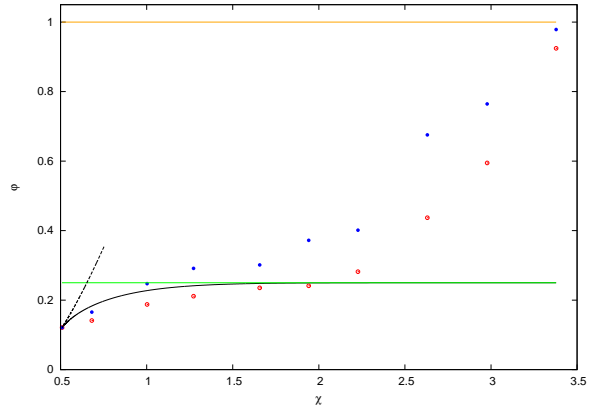

Fig. 2. *Solid curve: the relation between the density ratio (χ is its Briggs or decimal logarithm) and total-mass fraction (φ) for the case of Eq. (3), $r_2 \leq r_1$; dashed curve: the same, but for the reverse case, $r_1 \leq r_2$; two sets of points describe the same relation, but they correspond to two arbitrary special cases of the model extension. The horizontal asymptotes are at $\varphi=0.25$, for the solid curve, and at $\varphi=1$, which pertains to any of the two sets of points.*
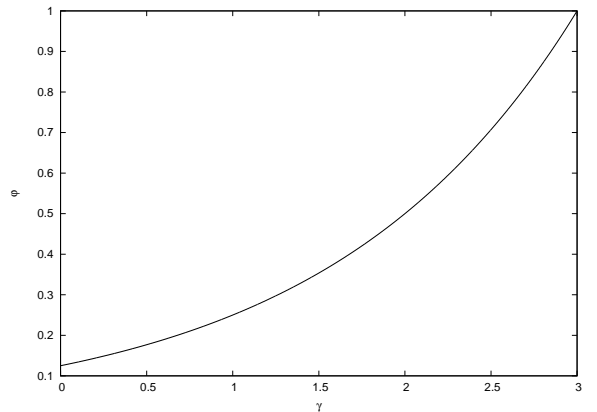

Fig. 3. *The dependence of the total-mass fraction on exponent γ according to Eq. (6).*

The total-mass fraction within the core radius can increase also under conditions of almost constant density within the core, as shown by the example discussed in Section 2 (the GIM model, version $r_2 \geq r_1$). The increase of the total-mass fraction within the core, as studied here, is due to the steepenning of the density profile within the core, quantitatively expressed through the ratio of the central density to that at the core radius.

A mass distribution where the density profile within the core is sufficiently steep is known as a cuspy one. Models (set of formulae) developed for the purpose of describing such distributions are known to exist (for the references see below). The extremal case of the model given by Eq. (1), $r_2=0$, as already said above (Section 1), belongs to a family of models dealing with a cuspy mass distribution. The property that in the periphery the density decreases following approximately the power law with exponent equal to 4 is valid for this family in general.

The usual way to express the cusp phenomenon is to assume that near the centre the density decreases following a power law. The consequence is that the central density becomes infinite (exception when the exponent equals zero) and therefore, the ratio of the central density to that at the core radius cannot be used as a steepness measure for the density profile within the core. Instead, one uses the exponent (more precisely, its modulus) of the power law applied near the centre.

In the case of the family of models (Dehnen 1993, Tremaine et al. 1994), the expression yielding the total-mass fraction within the core is very simple:

$$\varphi = \left(\frac{1}{2}\right)^{3-\gamma} , \qquad (6)$$

where γ is the exponent modulus for the central part; it satisfies the condition $0 \leq \gamma < 3$. The higher its value, the steeper is the density profile within the core. As easily seen, (Eq. (6) and Fig. 3), the total-mass fraction within the core becomes higher with increasing γ. Such a conclusion agrees well with the idea developed here, the steeper the density profile within the core, the higher is the total-mass fraction within the core. In other words the increase of the total-mass fraction within the core is due to the steepness of the density profile within the core, rather than to a faster density decrease in the envelope, beyond the core.

However, there is a difference. In the case of Eq. (6), the total-mass fraction of the envelope tends to zero when γ approaches 3. In the model considered here the density at any distance from the centre is represented as a sum of the contributions of two components. Though referred to as the inner and outer ones, these components should not be regarded as subsystems because they have no clear physical meaning. Their presence is vindicated by the circumstance that the unlimited increase of the mass of the inner component leads to both: the density ratio tends to infinity and the total-mass fraction within the core tends to be maximal (100%). Then the total mass of the outer component, which affects the envelope, does not itself tend to zero, it merely becomes arbitrarily small when compared to the total mass of the inner component. The role of the inner component is simply to explain the phenomenon of cuspy profiles.

Finally, the family of models (Dehnen 1993, Tremaine et al. 1994) does not also yield several possible trends which describe the increase of the total-mass fraction within the core. The steepenning of the density profile within the core, as well as the increase of the total-mass fraction as its consequence, is quantitatively expressed through the value of the parameter γ. Eq. (6) which relates γ to the total-mass fraction, yields only one value for the function for every argument value.

The family to which Eq. (6) belongs does, certainly, not exhaust the possibilities concerning the models with cusps. There are other models (e.g. Ossipkov Jiang 2007, also Raspopova et al. 2012 and the references therein). In most cases these density formulae appear as a special case of another formula (Zhao 1997 - Eq. (2)) which is sufficiently general. In particular, the density formula of Dehnen (1993) and Tremaine et al. (1994) is obtained by substituting $\alpha=1$ and $\beta=4$ for Zhao's parameters. The property of tending to zero for the total-mass fraction of the envelope when γ approaches 3 remains.

Among the models mentioned above only in the one presented here, the extremal cases occur simultaneously: the steepest density profile within the core, represented as infinite ratio of the central density to that at the core radius, and the maximal total-mass fraction of the core (100%).

5. CONCLUSION

Within the framework of the generalised isochrone model proposed by Kuzmin and Veltmann (1973) two branches are indicated here. One of them is further treated because of its useful property that the increase of ratio of the central density to that at the core radius is followed by the corresponding increase of the total-mass fraction within the core. In this way the cusp phenomenon is viewed more naturally, as a strong increase of the central density compared to that at the core radius.

Unfortunately, in this model the total-mass fraction within the core cannot exceed the value of 1/4 occuring when the density ratio tends to infinity. For this reason this model is here extended. After the extension, which provides the density to be calculated by adding two functions, the total-mass fraction within the core attains its maximal possible value of 100% when the density ratio tends to infinity.

Besides, the branch of the generalised isochrone model extended here yields a one-to-one correspondence of the total-mass fraction within the core to the density ratio because both quantities depend on the same parameter. It cannot be excluded that real stellar systems (subsystems) follow the relation between the total-mass fraction within the core and the density ratio, but only qualitatively, quantitatively they follow a few of them. The model extension proposed here is characterised by a larger number of parameters than the special case of the generalised isochrone model and, as a consequence, it offers the possibility of several particular relations between the total-mass fraction within the core and the density ratio - that at the centre to that corresponding to the core radius.

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ПОНОВНО РАЗМАТРАЊЕ МОДЕЛА РАСПОДЕЛЕ МАСЕ

S. Ninković

Astronomical Observatory, Volgina 7, 11060 Belgrade 38, Serbia
E–mail: *sninkovic@aob.rs*

УДК 524.7–333–34
Оригинални научни рад

Проучава се модел расподеле масе који су предложили Кузмин и Велтман (1973). Указује се на две његове варијанте које имају пресек. Само једна од њих је предмет проучавања. Нагласак се ставља на релацију између односа густина (густина у средишту према густини која одговара полупречнику средишњег дела) и удела укупне масе унутар средишњег дела. Друга величина је растућа функција прве, али она не може да буде већа од једне четвртине, а то се догађа када однос густина тежи бесконачности. Стога је модел проширен тако што се густина представља као збир две компоненте. После проширења добија се да када вредност односа густина тежи бесконачно, удео укупне масе тежи 100%. Број параметара проширеног модела већи је од истог за оригинални модел. Захваљујући овом кореспонденција између односа густине и удела укупне масе није више обострано једнозначна; више вредности удела укупне масе може да одговара истој вредности односа густина. На овај начин проширеним моделом би могла да се објасни могућност постојања две, или више, група стварних звезданих система (подсистема) на графику удео укупне масе у функцији од односа густина.