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FBILI METHOD FOR THE TWO-LEVEL ATOM LINE FORMATION IN MEDIA WITH LOW VELOCITY FIELDS

I. Pirković[1] and O. Atanacković[2]

[1]*Astronomical Observatory, Volgina 7, 11060 Belgrade 38, Serbia*
E–mail: *irena@aob.rs*

[2]*Department of Astronomy, Faculty of Mathematics, University of Belgrade Studentski trg 16, 11000 Belgrade, Serbia*
E–mail: *olga@math.rs*

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SUMMARY: In this paper we generalized the fast convergent Forth-and-Back Implicit Lambda Iteration (FBILI) method to the solution of the two-level atom line transfer problems in media with low velocity fields using the observer's reference frame. In order to test the accuracy and the convergence properties of the method we solved several astrophysically important benchmark problems of the NLTE line formation: in a plan-parallel differentially expanding medium of finite thickness, and in spherically symmetric stellar atmospheres, both static and expanding. We compared our solutions with those obtained by other authors using different numerical methods.

Key words. methods: numerical - radiative transfer - stars: atmospheres

1. INTRODUCTION

For modelling of many astrophysical objects it is necessary to solve the radiative transfer (RT) problem taking into account the motion of the medium. In media with low velocity regime such as solar prominences, circumstellar envelopes or shells, the dispersion of flow speeds is less than, or of the order of the mean thermal speed. In such cases the radiative transfer is usually solved in the observer's (laboratory) frame of reference. As in the static case, the RT equation 'along the ray' is an ordinary differential equation, but the opacity and emissivity of the material, as seen by the observer at rest, depend on the direction of propagation of radiation due to the Doppler effect. Angles and frequencies are coupled together by the Doppler shift. Using the observer's reference frame, most numerical techniques developed for static media can be straightforwardly applied to the RT in moving media with arbitrary (non-monotonic) velocity fields. Only a wider range of frequencies (due to Doppler shifts) and a larger number of angles (due to the coupling between the angle and frequency) must be used.

For strong flows, with speeds much larger than thermal (like e.g. expanding atmospheres of hot WR and Of stars and many early-type supergiants with strong stellar winds, novae and supernovae, accretion disks in close binary stars or in active galactic nuclei), radiative transfer is preferably formulated in the Co-Moving Frame (CMF) of reference (i.e. the frame comoving with the fluid). The Lagrangian equation of RT in moving media has been successfully solved in various astrophysical problems. The basic disadvantage of CMF calculations is the imposition of monotonic velocity fields. Baron and Hauschildt (2004) presented an operator splitting method to solve 1D spherical RT for arbitrary velocity fields in the CMF, whereas Knop et al. (2009) proposed a new formal solution of the RT, which avoids the negative opacities occurring in the non-monotonic flows. In the study of high-speed outflows from stars, supernovae etc., where the velocity gradients greatly enhance the escape of photons, Sobolev (or the large-velocity gradient - LVG) approximation is commonly used (Sobolev 1957). To model such strong flows, it is necessary to solve the RT simultaneously with hydrodynamic equations. Such problems require ever more efficient, exact and fast convergent numerical methods.

The earliest numerical solutions of RT in moving media were those by Chandrasekhar (1945), who solved the plane-parallel Schuster problem and the planetary nebula Lyman α problem in an expanding slab. Most of the later work on moving atmospheres was based on generalization of the Feautrier method (1964), restricted first to planar geometry. Kunasz and Hummer (1974a) obtained solutions to the line transfer problem in expanding spherical atmospheres using the variable Eddington factor method with the Rybicki elimination scheme. Kunasz and Hummer (1974b) and Mihalas, Kunasz and Hummer (1975, hereafter MKH) employed a "direct solution" and ray technique in which the radiative transfer is considered along each of many parallel rays contributing to a set of ordinary differential equations coupled by the scattering integral. Avrett and Loeser (1984, hereafter AL) proposed a method which uses the integral form of the RT equation along the set of rays. Rogers (1984) proposed the half-range moment method of solving the RT problem in spherical geometry which, however, was restricted to the CMF. In the last three decades fast convergent and efficient, so-called ALI (Accelerated Lambda Iteration) methods were developed and applied to the solution of RT in moving media (for a review see e.g. Hamann 1987, Hamann 2003).

In this paper we solve the line formation problem in moving (both plane-parallel and spherically symmetric) media by the use of the Forth-and-Back Implicit Lambda Iteration (FBILI) method developed by Atanacković-Vukmanović, Crivellari and Simonneau (1997; hereinafter ACS97). This method proved to be exact, fast convergent and extremely efficient when applied to line formation (both by two-level and multilevel atoms) in 1D static media (ACS97) and by two-level atoms in 2D static media (Milić and Atanacković 2014). For comparison of its convergence properties with respect to other ALI methods see also the paper by Atanacković-Vukmanović (2007). Here, we intend to generalize it to the two-level atom line transfer in 1D moving media and to examine its properties using the observer's reference frame. The solution in the Co-Moving Frame of reference will be given in a forthcoming paper.

The outline of the paper is as follows. In Section 2 we present the basic equations of the two-level atom line transfer problem in a spherically symmetric moving atmosphere and, in Section 3, we describe in detail the method proposed for its solution. In Section 4 we solve several astrophysically interesting benchmark problems given by Hummer and Rybicki (1968) and Avrett and Loeser (1984) and discuss the obtained numerical results.

2. LINE TRANSFER IN MOVING MEDIA (IN THE OBSERVER'S FRAME)

Let us consider in detail the case of a two-level atom line formation in a spherically symmetric expanding stellar atmosphere. Transition to the case of a plane-parallel expanding slab of finite thickness is quite straightforward, and will be discussed in Section 4. We shall assume that physical properties vary with only one coordinate - the radial distance r. To describe the velocity effects we shall use the observer's frame of reference.

For a spherically symmetric, radially expanding atmosphere, the radiative transfer equation (RTE) in the observer's frame takes the following form:

$$\begin{aligned}\mu\ \frac{\partial I(r,\nu,\mu)}{\partial r}+\frac{1-\mu^2}{r}\ \frac{\partial I(r,\nu,\mu)}{\partial \mu}= \\ -\chi(r,\nu,\mu)\left[I(r,\nu,\mu)-S(r,\nu,\mu)\right].\end{aligned} \tag{1}$$

Here, $I(r,\nu,\mu)$ is the specific intensity at point r, at frequency ν and in direction μ (cosine of the angle θ between the local outward radial direction and the direction of propagation of radiation at radius r). In moving media, the absorption coefficient $\chi(r,\nu,\mu)$ and the source function $S(r,\nu,\mu)$, as seen by the observer at rest, depend on the direction of propagation of radiation. Let us remind that they are isotropic only in static media with isotropic scattering. When the atmospheric gas moves with velocity $\vec{v}(r)$ with respect to the observer, the angular dependence of opacity and emissivity of the material in the observer's frame is caused by the Doppler shift between the frequency in the observer's frame ν and in the material rest frame ν':

$$\nu'=\nu-\frac{\nu_0}{c}\vec{l}\cdot\vec{v}\ . \tag{2}$$

Here, ν_0 is the central line frequency in the laboratory frame and $\vec{l}$ is the direction of photons' propagation. It is convenient to use the dimensionless frequency $x=(\nu-\nu_0)/\Delta\nu_D^*$, i.e. the frequency displacement from the line center expressed in standard Doppler width units $\Delta\nu_{\rm D}^*=\nu_0 v_{\rm th}^*/c$. Here, $v_{\rm th}^*$ is the mean thermal velocity at some standard temperature T^*. The velocities are measured in the same units $V=v/v_{\rm th}^*$, so that the relation (2) becomes:

$$x'=x-\mu V\ . \tag{3}$$

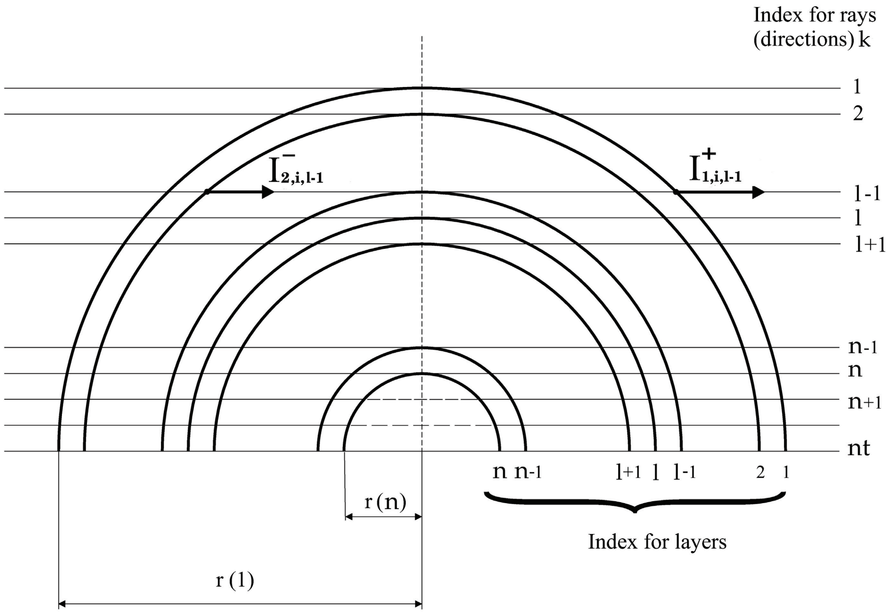

Fig. 1. *Discrete mesh of radii* $\{r_l\}, l = 1, n$ *and a grid of rays (directions)* $z_k; k = 1, nt$ *that are used for the solution of the RT equation;* $I^{\pm}_{l,i,k}$ *denote the in–coming and out–going specific intensities at frequency i along the direction k at any point l.*

Instead of solving the RTE as a partial differential equation (1), we can perform a ray-by-ray computation of the specific intensities along the set of directions tangent to the spherical layers (like those shown in Fig. 1) using the ordinary differential RTE in the 'along the ray' form:

$$\pm\frac{dI^{\pm}(x,\mu)}{d\tau(x,\mu)} = I^{\pm}(x,\mu) - S(x,\mu) \; . \qquad (4)$$

Here, τ represents the optical distance along a given direction (ray) measured from the surface, whereas $I^{\pm}$ are the specific intensities in two directions along the ray. According to the customary terminology and notation, the intensity propagating in the direction of increasing optical depth along the ray (in-going intensity) is denoted as I^{-}, while that propagating in the direction of decreasing optical depth (out-going intensity) is denoted by I^{+}. Since all the quantities are depth dependent, in Eq. (4) we omitted their spatial dependence, and we shall not refer to it unless it is needed.

The monochromatic optical distance along a given direction is given by

$$d\tau(x,\mu) = -\chi(x,\mu)dz \, , \qquad (5)$$

where dz is the related geometrical path length and the volume opacity coefficient χ generally contains the corresponding line and continuum contributions. The velocity fields have no significant influence on the RT in the continuum. On the contrary, the line formation can be affected a lot as even a small Doppler shift gives rise to large changes in line absorption as seen by the observer. Thus, the opacity is affected by motion through the line profile and can be written as:

$$\chi(x,\mu) = \chi^{\rm C} + \chi^{\rm L}\phi(x,\mu) \, , \qquad (6)$$

where $\chi^{\rm C}$ and $\chi^{\rm L}$ are the continuum and mean line absorption coefficient, respectively, and ϕ is the normalized line profile function given by:

$$\phi(x,\mu) = \phi(x-\mu V) \, . \qquad (7)$$

For pure Doppler broadening the corresponding Gaussian profile function is

$$\phi(x,\mu) = \frac{1}{\delta\sqrt{\pi}} e^{-(x-\mu V)^2/\delta^2} \, , \qquad (8)$$

where $\delta = \Delta\nu_{\rm D}/\Delta\nu^{*}_{\rm D}$ is the ratio of the Doppler widths at a local temperature and at some standard temperature T^{*}. The profile is no longer symmetric and it has to be estimated for each angle-velocity dependent frequency point from the interval $(-x-\mu V, x+\mu V)$.

If we denote the ratio of the continuum to the mean line absorption coefficient with $\beta = \chi^{\rm C}/\chi^{\rm L}$, using Eq. (6) we can write Eq. (5) as follows:

$$d\tau(x,\mu) = d\tau^{\rm L}\left[\beta + \phi(x,\mu)\right] \, , \qquad (9)$$

where

$$d\tau^{\rm L} = -\chi^{\rm L}(z)dz \qquad (10)$$

is the mean line optical depth scale along the ray.

Finally, the total (continuum + line) source function $S(x,\mu)$, defined as the ratio of the total emission coefficient $\eta(x,\mu)$ to the total absorption coefficient $\chi(x,\mu)$, is given by:

$$S(x,\mu) = \frac{\eta^{\mathrm{C}} + \eta^{\mathrm{L}}\psi(x,\mu)}{\chi^{\mathrm{C}} + \chi^{\mathrm{L}}\phi(x,\mu)} \ . \tag{11}$$

The emission profile function $\psi(x,\mu)$ generally differs from the absorption one $\phi(x,\mu)$ due to the line frequency redistribution. For simplicity, we will assume that the absorptions and re-emissions are completely uncorrelated and that the two profiles are equal ($\psi = \phi$). This approximation of the complete frequency redistribution (CRD) is inconsistent in a moving medium and the angle-dependent partial frequency redistribution (PRD) of photons in line should be considered. This will be taken into account together with the solution of RT in the comoving frame in a forthcoming paper. Using the expressions $S^{\mathrm{L}} = \eta^{\mathrm{L}}/\chi^{\mathrm{L}}$ and $S^{\mathrm{C}} = \eta^{\mathrm{C}}/\chi^{\mathrm{C}}$ for the line and the continuum source function, respectively, the total source function (11) can be written as:

$$S(x,\mu) = \frac{\beta}{\beta + \phi(x,\mu)} S^{\mathrm{C}} + \frac{\phi(x,\mu)}{\beta + \phi(x,\mu)} S^{\mathrm{L}} \ . \tag{12}$$

For the continuum source function the solution of the monochromatic scattering problem in a static medium can be applied. The line source function generally depends on the radiation field, which is strongly influenced by the motion of the gas it interacts with. The line source function for a two-level atom under the assumption of CRD is given by (see Mihalas 1978):

$$S^{\mathrm{L}} = \varepsilon B + (1-\varepsilon)\bar{J} \ , \tag{13}$$

where ε is the photon destruction probability if the stimulated emission term is omitted, B is the Planck function and

$$\bar{J} = \frac{1}{2}\int_{-\infty}^{\infty} dx \int_{-1}^{1} d\mu I(x,\mu)\phi(x,\mu) \tag{14}$$

is the angle and line profile integrated intensity (the so-called scattering integral). Note that all the specific intensities $I(x,\mu)$, i.e. all the specific RTE (4) are coupled by this scattering term.

Once the line formation problem is defined by Eq. (4) and Eqs. (12) - (14), we can look for its numerical solution.

2.1. Discretized form of the problem equations

For the numerical description of the radiation transport through 1D moving spherical atmosphere, a discrete set of radii $\{r_l\}$, $l = 1, n$ is needed (Fig. 1). Let the radii r_1 and r_n correspond to the upper and lower boundary surfaces of the atmosphere, respectively. We take r_1 as the origin of the mean line optical depth scale along the radial direction, defined by:

$$\tau^{\mathrm{L}}(r) = \int_r^{r_1} \chi^{\mathrm{L}}(r')dr' \ . \tag{15}$$

Here, since we are considering a two-level atom line formation, the mean line opacity $\chi^{\mathrm{L}}(r)$ is assumed to be known, so that we can compute the set of mean line radial optical depths $\tau_l = \tau(r_l)$ starting from $\tau_1 = 0$.

The solution of RTE (4) is performed along the set of rays $\{z_k\}, k = 1, n$ tangent to the spherical layers corresponding to the discrete set of radii $\{r_l\}$, as well as along a few additional, so-called core rays $\{z_k\}, k = n+1, nt$ that intersect the inner boundary surface (see Fig. 1). Hence, for a set of line frequencies $\{x_i\}, i = 1, nf$ and rays $\{z_k\}, k = 1, nt$, Eq. (4) can be written in the discretized form:

$$\pm\frac{dI^{\pm}_{i,k}}{d\tau_{i,k}} = I^{\pm}_{i,k} - S_{i,k} \ . \tag{16}$$

The values of specific intensities and source functions are to be computed at points where the ray intersects spherical shells of radii r_l with $l < k$. Let us denote these values as $I^{\pm}_{l,i,k}$ and $S_{l,i,k}$, respectively, for $l = 1, n$. At these points l, the ray k forms angles $\theta_{l,k}$ with the local outward radial directions whose cosines $\mu_{l,k}$ are given by

$$\mu_{l,k} = \pm\sqrt{1 - \frac{r_k^2}{r_l^2}} \ . \tag{17}$$

The + sign stands for outgoing ($\mu > 0$), and - sign for ingoing rays ($\mu < 0$). In Eq. (16), $d\tau_{i,k}$ is the optical path length between two layers $l-1$ and l along the ray k at frequency i:

$$d\tau_{i,k} = -\chi_{l,i,k}(z)dz \ , \tag{18}$$

where $dz = dr/\mu(r)$ and the absorption coefficient is given by:

$$\chi_{l,i,k} = \chi^{\mathrm{L}}_l[\beta_l + \phi_{l,i,k}] \ . \tag{19}$$

If we assume, for simplicity, an isothermal medium ($\delta(z) = 1$), the profile function $\phi_{l,i,k}$ (see Eq. (8)) becomes:

$$\phi_{l,i,k} = \frac{1}{\sqrt{\pi}} e^{-(x_i - \mu_{l,k}V_l)^2} \ . \tag{20}$$

The mesh size on frequency should be large enough to include the whole line profile which can vary over the range $\pm V_{\max}$. Usually, $V < 0$ is used for motions towards and $V > 0$ for motions away from the observer. In the test problem considered in Section 4, the convention is opposite.

The total source function (12) can be written in the discretized form:

$$S_{l,i,k} = \frac{\beta_l}{\beta_l + \phi_{l,i,k}} S^{\mathrm{C}}_l + \frac{\phi_{l,i,k}}{\beta_l + \phi_{l,i,k}} S^{\mathrm{L}}_l \ . \tag{21}$$

with the line source function

$$S^{\mathrm{L}}_l = \varepsilon_l B_l + (1-\varepsilon_l)\bar{J}_l \tag{22}$$

and the scattering integral (14) replaced by a finite sum of specific intensities

$$\bar{J}_l = \frac{1}{2}\sum_{i=1}^{nf}\sum_{k=l}^{nt} w_{l,i,k}[\phi^+_{l,i,k} I^+_{l,i,k} + \phi^-_{l,i,k} I^-_{l,i,k}] \; . \quad (23)$$

with the quadrature weights $w_{l,i,k}$ for the integration over directions and line frequencies, that satisfy the condition $\sum_{i=1}^{nf}\sum_{k=l}^{nt} w_{l,i,k} = 1$. The profile functions $\phi^+_{l,i,k}$ and $\phi^-_{l,i,k}$ correspond to the two opposite projections of the gas velocity $\pm\mu_{l,k}V_l$ along the ray k, at point l.

3. SOLUTION OF THE LINE TRANSFER IN MOVING MEDIA (IN THE OBSERVER'S FRAME)

Here we shall consider the solution of the problem equations (16) and (21) describing the two-level atom line transfer with complete frequency redistribution in 1D moving isothermal medium with a frequency independent continuum source function.

The simplest way to solve not only this but any non-LTE problem is by means of Λ iteration that solves the radiative transfer equation

$$\bar{J} = \Lambda S \quad (24)$$

and statistical equilibrium (SE) equations $S = S(\bar{J})$ in turn. However, in most cases of interest (large departures from LTE and for large optical depths) its convergence is extremely slow.

Before we describe the way in which the FBILI method solves the above problem, let us briefly recall the idea that is in the heart of a broad class of the so-called ALI (Accelerated Lambda Iteration) methods.

A way to significantly accelerate the convergence of the Λ iteration while retaining its simplicity, is to replace the exact Λ operator by an approximate one and to correct the error introduced by this approximation iteratively. The approximations should enable a better conditioning of the problem equations. This idea of operator splitting was introduced in the RT computations by Cannon (1973a,b). The full (exact) Λ operator is split into the Approximate Lambda Operator (ALO), Λ^*, which is easy to invert, and the part $(\Lambda - \Lambda^*)$ that can be treated as a perturbation:

$$\Lambda = \Lambda^* + (\Lambda - \Lambda^*). \quad (25)$$

By substituting the above expression into Eq. (24) we get:

$$\bar{J} = \Lambda S = \Lambda^* S + (\Lambda - \Lambda^*)S. \quad (26)$$

If, at the moment, we neglect the processes in the background continuum and combine Eq. (26) with the SE equation (13), we obtain a simple iterative procedure. Olson et al. (1986) showed that the diagonal of the full Λ matrix is the most optimal ALO. In that case, the inversion of the Λ^* matrix is replaced by a simple division:

$$S^{i+1} = \frac{\varepsilon B + (1-\varepsilon)(\Lambda - \Lambda^*)S^i}{1-(1-\varepsilon)\Lambda^*} \; . \quad (27)$$

This ALI method, known as the Jacobi method, accelerates the convergence of the classical Λ iteration by several orders of magnitude. Because of its simplicity and much higher convergence rate the Jacobi method is the most often used ALI method. However, in practice, even this method needs to be further accelerated by some mathematical techniques as, e.g., using the Ng acceleration (Ng 1974).

In this paper, we apply the FBILI method to solve the above mentioned line transfer problem more efficiently than with the Jacobi method using no additional mathematical acceleration technique. As the analytical solution of this problem can not be obtained, to investigate the accuracy of the FBILI procedure we solve several benchmark problems (see Section 4) and we compare our results with those obtained by other authors. Since we have not found in the literature the convergence rate of other iterative methods used to solve the same benchmark problems, we here use the solutions obtained by our version of the Jacobi method as the reference ones with no additional acceleration. Namely, for the sake of comparison, we apply to the Jacobi method the same formal solution used by the FBILI. In Subsection 3.1 we describe the formal solution and our variant of the Jacobi method in detail, whereas in Subsection 3.2. we expose the basic ideas and equations used in the FBILI method.

3.1. Formal solution with short characteristics in two-points and the Jacobi-type iterative procedure

Although the radiation field is unknown, using the two-stream approximation we can represent its propagation by means of integral form of the RTE for both the in-going and the out-going specific intensities, as follows

$$I_l = I_{l-1}e^{-\Delta} + \int_0^{\Delta} S(t)e^{t-\Delta}dt \; . \quad (28)$$

In the standard short characteristics approach (e.g. Kunasz and Auer 1988) to the formal solution, the variation of the source function with the optical depth along the ray is assumed to be a parabola between three consecutive grid points. In the FBILI method (see ACS97) and in our variant of the Jacobi method that will be presented here, we will assume a parabolic behavior of the source function between two successive depth points. Proceeding in this way we will derive the implicit linear relation between the mean intensity of the radiation field and the local line source function:

$$\bar{J} = a + bS^{\mathrm{L}} \; . \quad (29)$$

This relation is implicit as the value of the source function is also unknown. It depends on the unknown radiation field via scattering processes. One can see that Eq. (29) corresponds to Eq. (26) of ALI methods, with the coefficient b playing a role of the diagonal ALO in the Jacobi method. By substituting Eq. (29) into SE equation (13), we get the expression similar to Eq. (27) for updating the source function

$$S^{\mathrm{L}} = \frac{\varepsilon B + (1-\varepsilon)a}{1-(1-\varepsilon)b} \ . \tag{30}$$

Briefly, during the formal solution we can compute the coefficients a and b of the linear relation (29) and then use them in Eq. (30) to update the line source function S^{L} at all depth points.

The iterative computation of these coefficients and not of the unknown functions ($\bar{J}$ and S^{L}) themselves like in the classical Λ iteration, speeds up the convergence dramatically. The unknowns, $\bar{J}$ and S^{L}, change in a similar way and, consequently, the coefficients a and b of their linear relation change very little from one iteration to another. As good quasi-invariants of the problem, they attain very quickly their exact values and lead very fast to the exact solution of the whole procedure.

Now, we shall describe in more detail the formal solution performed by the use of short characteristics in two successive depth points, as it will be used in both the Jacobi-type and the FBILI solution.

3.1.1. Forward step

To compute the in-going intensities along the ray $I^-_{l,i,k}$, i.e. to compute the coefficients a^-_l and b^-_l of the implicit linear relation corresponding to inward directions, we start with the given values of the specific intensities at the upper boundary surface of an atmosphere. At $\tau_1 = 0$ we assume $I^-_{1,i,k} = 0$ for all frequencies i and along all rays k and, hence, we put $a^-_1 = b^-_1 = 0$. The propagation of the unknown in-going intensities can be represented by means of the integral form of the RT equation:

$$I^-_{l,i,k} = I^-_{l-1,i,k} e^{-\Delta} + \int_0^{\Delta} S(t) e^{t-\Delta} dt \ , \tag{31}$$

where:

$$\Delta = \Delta^{l-1,l}_{i,k} = \int_{z_{l-1,k}}^{z_{l,k}} \chi_{i,k}(z) dz = \int_{r_{l-1}}^{r_l} \frac{\chi_{i,k}(r')dr'}{\mu_k(r')} \tag{32}$$

is the monochromatic optical path between two points $l-1$ and l along the ray k. S is the total source function given by Eq. (21).

To compute the in-going specific intensities at all subsequent points $l = 2, n$, we use some polynomial approximation for the source function between two successive depth points $l-1$ and l. Here, a piecewise parabolic behavior for the source function is assumed, and the derivatives at two limiting points of the interval $(l-1, l)$ are then related by:

$$S'_{l-1,i,k} = \frac{2}{\Delta}[S_{l,i,k} - S_{l-1,i,k}] - S'_{l,i,k} \ . \tag{33}$$

By solving the integral in Eq. (31) by parts and using Eq. (33), we can write Eq. (31) as follows:

$$I^-_{l,i,k} = I^-_{l-1,i,k} e^{-\Delta} + \mathcal{Q}^- S_{l-1,i,k} + \mathcal{P}^- S_{l,i,k} + \mathcal{R}^- S'_{l,i,k} \ . \tag{34}$$

The coefficients $\mathcal{P}^-$, $\mathcal{Q}^-$ and $\mathcal{R}^-$ depend solely on Δ and have the following form:

$$\mathcal{Q}^- = \frac{2}{\Delta^2} - e^{-\Delta}\left(1 + \frac{2}{\Delta} + \frac{2}{\Delta^2}\right)$$

$$\mathcal{P}^- = 1 - \frac{2}{\Delta^2} + e^{-\Delta}\left(\frac{2}{\Delta} + \frac{2}{\Delta^2}\right)$$

$$\mathcal{R}^- = -1 + \frac{2}{\Delta} - e^{-\Delta}\left(1 + \frac{2}{\Delta}\right) \ . \tag{35}$$

They are to be computed at each depth point l, for each frequency i and along each ray k. Introducing a new variable for the non-local terms

$$\widetilde{\mathcal{Q}}^- = I^-_{l-1,i,k} e^{-\Delta} + \mathcal{Q}^- S_{l-1,i,k} \ , \tag{36}$$

and keeping the source function derivative (that contains both the local and non-local terms), Eq. (34) can be written as:

$$I^-_{l,i,k} = \widetilde{\mathcal{Q}}^- + \mathcal{P}^- S_{l,i,k} + \mathcal{R}^- S'_{l,i,k} \ . \tag{37}$$

Let us remind here that the classical Λ iteration uses Eq. (34), i.e. Eq. (37), to compute the in-going specific intensities at all depth points from the old (obtained in the previous iteration) values of the source function and its derivatives.

In the Jacobi method we proceed in a different way. We want to obtain a linear relation between the in-going mean intensity $\bar{J}^-_l$ and the local line source function S^{L}_l at each depth point l:

$$\bar{J}^-_l = a^-_l + b^-_l S^{\mathrm{L}}_l \ . \tag{38}$$

It means that we have to substitute the expression for the total source function (21) and its derivative $S'_{l,i,k}$ in Eq. (37), to separate the term with the local line source function from the rest, and to integrate Eq. (37) over frequencies and directions. The coefficients of Eq. (38) are then given as:

$$a^-_l = \frac{1}{2}\sum_{i=1}^{nf}\sum_{k=l}^{nt}\left\{\widetilde{\mathcal{Q}}^- + \mathcal{P}^-\left(\frac{\beta}{\phi+\beta}\right)S^{\mathrm{C}} + \mathcal{R}^-\left(\frac{\mu}{\phi+\beta}\right)\left[\left(\frac{\beta S^{\mathrm{C}}_l}{\phi+\beta}\right)' + \left(\frac{\phi}{\phi+\beta}\right)S'^{\mathrm{L}}\right]\right\} w_{l,i,k}$$

and

$$b_l^- = \frac{1}{2}\sum_{i=1}^{nf}\sum_{k=l}^{nt}\Big[\mathcal{P}^-\left(\frac{\phi}{\phi+\beta}\right) + \mathcal{R}^-\left(\frac{\mu}{\phi+\beta}\right)\left(\frac{\phi}{\phi+\beta}\right)'\Big] w_{l,i,k} \ . \tag{39}$$

Let us note that the derivatives in the above coefficients are the derivatives with respect to the radial mean line optical depth. They are computed by using the Lagrangean polynomial of the second degree in three points. The values of intensities needed to get the coefficients $\widetilde{\mathcal{Q}}^-$, are obtained by recursive application of relation (34) with the old values of S and S', proceeding from the given boundary condition at the surface.

For variables other than $w_{l,i,k}$, the indices l, i and k are left out for simplicity. Let us, however, remind that β, S^{C} and S^{L} depend on optical depth only, while $\phi, \Delta, S, \mathcal{P}^-, \widetilde{\mathcal{Q}}^-, \mathcal{R}^-$ are also angle and frequency dependent.

3.1.2. Backward step

Now, we proceed from the bottom layer where the out-going specific intensities are known. For the rays with $k > n$ we use the diffusion approximation or we simply take that $I^+_{n,i,k>n} = S_{n,i,k}$, whereas for $k = n$ the condition $I^+_{n,i,n} = I^-_{n,i,n}$ is to be used. At all other upper points $l = n-1, 1$, we can compute $I^+_{l,i,k}$ using the integral form of the RT equation:

$$I^+_{l,i,k} = I^+_{l+1,i,k}e^{-\Delta} + \int_0^\Delta S(t)e^{t-\Delta}dt \ . \tag{40}$$

Here, we again approximate the source function between two successive depth points by a parabola to get:

$$I^+_{l,i,k} = I^+_{l+1,i,k}e^{-\Delta} + \mathcal{Q}^+S_{l+1,i,k} + \mathcal{P}^+S_{l,i,k} + \mathcal{R}^+S'_{l+1,i,k} \ , \tag{41}$$

where:

$$\mathcal{Q}^+ = \frac{2}{\Delta} - \frac{2}{\Delta^2} - e^{-\Delta}\left(1 - \frac{2}{\Delta^2}\right)$$

$$\mathcal{P}^+ = 1 - \frac{2}{\Delta} + \frac{2}{\Delta^2} - e^{-\Delta}\frac{2}{\Delta^2}$$

$$\mathcal{R}^+ = -1 + \frac{2}{\Delta} - e^{-\Delta}\left(1 + \frac{2}{\Delta}\right) \ . \tag{42}$$

Let us remind here again that the classical Λ iteration computes the out-going specific intensities $I^+_{l,i,k}$ at all depth points with the old values of the source function S and its derivatives S'.

To apply the Jacobi method, we have to separate the local term from the rest of Eq. (41) and rewrite it in the following form:

$$I^+_{l,i,k} = \widetilde{\mathcal{Q}}^+ + \mathcal{P}^+S_{l,i,k} \ . \tag{43}$$

Here, we introduced the coefficient:

$$\widetilde{\mathcal{Q}}^+ = I^+_{l+1,i,k}e^{-\Delta} + \mathcal{Q}^+S_{l+1,i,k} + \mathcal{R}^+S'_{l+1,i,k}$$

and computed it with the old values of S and S', while I^+ is obtained by the recursive application of Eq. (41).

Substituting the expression (21) for the total source function $S_{l,i,k}$ in Eq. (43) and integrating Eq. (43) over line profile frequencies and directions, we get a linear relation:

$$\bar{J}^+_l = a^+_l + b^+_l S^{\mathrm{L}}_l \ . \tag{44}$$

The coefficients are given as:

$$a^+_l = \frac{1}{2}\sum_{i=1}^{nf} I^-_{l,i,l}w_{l,i,l} + \frac{1}{2}\sum_{i=1}^{nf}\sum_{k=l+1}^{nt}\Big(\widetilde{\mathcal{Q}}^+ + \mathcal{P}^+\frac{\beta}{\phi+\beta}S^{\mathrm{C}}_l\Big) w_{l,i,k}$$

and

$$b^+_l = \frac{1}{2}\sum_{i=1}^{nf}\sum_{k=l+1}^{nt}\mathcal{P}^+\frac{\phi}{\phi+\beta}\omega_{i,k} \ , \tag{45}$$

where we used that $I^+_{l,i,l} = I^-_{l,i,l}$ for $k = l$. By summing up the coefficients given by equations (39) and (45), we obtain the total coefficients $a_l = a^+_l + a^-_l$ and $b_l = b^+_l + b^-_l$ of the linear relation (29) at all depth points $l = 1, n$. The line source function is then updated by means of Eq. (30). Once we have the new value of S^{L}_l we can obtain the total source function $S_{l,i,k}$ using Eq. (21). For the continuum source function S^{C}_l we use the solution of the monochromatic scattering case (for the details see Atanacković-Vukmanović 2003).

The computation of coefficients a_l and b_l, and of the updated line source function S^{L}_l is performed in turn until some prescribed convergence criterion is satisfied.

This small modification of the classical Λ iteration, consisting in the iterative computation of the coefficients of the linear relation between $\bar{J}$ and S^{L}, significantly accelerates the iterative procedure.

In the next subsection we shall describe how the FBILI method accelerates it even more.

3.2. FBILI solution

The FBILI solution differs from the previously described procedure in a few important points.

In the first (forward) part of each iteration, we compute the coefficients of an "in-going" linear relation retaining implicitly not only the local source function but also its derivative:

$$\bar{J}^-_l = \tilde{a}^-_l + \tilde{b}^-_l S^{\mathrm{L}}_l + c^-_l S'^{\mathrm{L}}_l \ . \tag{46}$$

The coefficients of the above relation are given by:

$$\tilde{a}_l^- = \frac{1}{2}\sum_{i=1}^{nf}\sum_{k=l}^{nt}\left[\widetilde{\mathcal{Q}}^- + \mathcal{P}^-\frac{\beta}{\phi+\beta}S_l^{\rm C} + \mathcal{R}^-\left(\frac{\mu}{\phi+\beta}\right)\left(\frac{\beta S_l^{\rm C}}{\phi+\beta}\right)'\right]w_{l,i,k}\ ,$$

$$\tilde{b}_l^- = \frac{1}{2}\sum_{i=1}^{nf}\sum_{k=l}^{nt}\left[\mathcal{P}^-\frac{\phi}{\phi+\beta} + \mathcal{R}^-\left(\frac{\mu}{\phi+\beta}\right)\left(\frac{\phi}{\phi+\beta}\right)'\right]w_{l,i,k}\ ,$$

and:

$$c_l^- = \frac{1}{2}\sum_{i=1}^{nf}\sum_{k=l}^{nt}\mathcal{R}^-\left(\frac{\mu}{\phi+\beta}\right)\frac{\phi}{\phi+\beta}w_{l,i,k}\ . \tag{47}$$

Furthermore, we introduce the ratio of the non-local part of the radiation field and the local line source function $\tilde{a}_l^-/S_l^{\rm L}$ as the iteration factor in the "local" coefficient b_l^-:

$$b_l^- = \frac{\tilde{a}_l^-}{S_l^{\rm L}} + \tilde{b}_l^-\ , \tag{48}$$

and get the following implicit linear relation between $\bar{J}_l^-$ and the line source function $S^{\rm L}$ and its derivative $S_l'^{\rm L}$:

$$\bar{J}_l^- = b_l^- S_l^{\rm L} + c_l^- S_l'^{\rm L}\ . \tag{49}$$

Defined as the ratio of two homologous quantities, the iteration factor quickly reaches its exact value speeding up the convergence of the whole iterative procedure.

The third important difference is introduced in the backward process. In order to take advantage of the known values of the coefficients a and b as soon as they are available to update the values of the source function, in the FBILI method we proceed as follows. We start from the lower boundary condition where the out-going specific intensities are known and, therefore, the coefficients of the linear relation between $\bar{J}_n^+$ and $S_n^{\rm L}$:

$$\bar{J}_n^+ = a_n^+ + b_n^+ S_n^{\rm L}\ ,$$

where:

$$a_n^+ = \frac{1}{2}\sum_{i=1}^{nf} I_{n,i,n}^- w_{n,i,n} + \frac{1}{2}\sum_{i=1}^{nf}\sum_{k=n+1}^{nt}\left[\left(\frac{\beta}{\phi+\beta}S_n^{\rm C} + S_{n,i,k}'\right)\right]w_{n,i,k}$$

and

$$b_n^+ = \frac{1}{2}\sum_{i=1}^{nf}\sum_{k=n+1}^{nt}\frac{\phi}{\phi+\beta}w_{n,i,k}\ . \tag{50}$$

Using the coefficients b_n^- and c_n^- in Eq. (49), computed and stored in the forward process, and assuming $S_n'^{\rm L} = 0$, we can obtain the coefficients of the relation:

$$\bar{J}_n^- = a_n^- + b_n^- S_n^{\rm L},$$

i.e. we take $a_n^- = 0$ and b_n^- as in Eq. (48).

With the given a_n^+ and b_n^+, we obtain the total coefficients a_n and b_n. Substituting $\bar{J}_n = a_n + b_n S_n^{\rm L}$ into the SE equation (22), we get the updated line source function at depth point $l = n$:

$$S_n^{\rm L} = \frac{\varepsilon B + (1-\varepsilon)a_n}{1-(1-\varepsilon)b_n} \tag{51}$$

and, thus, the updated values of the total source function:

$$S_{n,i,k} = \frac{\beta}{\phi+\beta}S_n^{\rm C} + \frac{\phi}{\phi+\beta}S_n^{\rm L}\ , \tag{52}$$

its derivatives $S_{n,i,k}'$ and the new values of the outgoing intensities:

$$I_{n,i,k}^+ = S_{n,i,k} + S_{n,i,k}'\ . \tag{53}$$

With these new values of $S_{n,i,k}$, $S_{n,i,k}'$ and $I_{n,i,k}^+$, we obtain a new value of $\widetilde{\mathcal{Q}}^+$ in Eq. (43), and new coefficients a_l^+ and b_l^+ of Eq. (44) at the next upper depth point l.

For all other upper depth points we proceed as follows. By substituting the expression

$$S_l'^{\rm L} = 2\left(S_{l+1}^{\rm L} - S_l^{\rm L}\right)/\Delta\tau_{l,l+1} - S_{l+1}'^{\rm L} \tag{54}$$

similar to Eq. (33), which is now written for $S^{\rm L}$ at points l and $l+1$, into Eq. (49) we get

$$a_l^- = c_l^-\left(\frac{2S_{l+1}^{\rm L}}{\Delta\tau_{l,l+1}} - S_{l+1}'^{\rm L}\right)$$

$$b_l^- = b_l^- - 2c_l^-/\Delta\tau_{l,l+1}\ , \tag{55}$$

where $\Delta\tau_{l,l+1}$ is the radial optical distance between the two points l and $l+1$.

Upon getting the total coefficients $a_l = a_l^+ + a_l^-$ and $b_l = b_l^+ + b_l^-$ at depth point l we can immediately get the new value of the source function applying Eq. (30). With this new line source function, we get the new values of the total source function and of its derivatives. Then we can compute the outgoing intensities $I_{l,i,k}^+$ by means of Eq. (41). We then proceed to the next layer and repeat the procedure until we reach the surface ($l = 1$). The procedure is repeated until the convergence criterion is satisfied.

4. TEST PROBLEMS AND RESULTS

To test the FBILI method when applied to the RT in moving media, we solved several benchmark problems.

4.1. Line formation in a plane-parallel moving slab

First, we solved the problem of the RT in a plane-parallel expanding slab of finite thickness (Hummer and Rybicki 1968). The center of the slab is at rest whereas the part of the slab closer to the observer is moving towards and the part that is farther is moving away from the observer with the velocity normal to the surface. There is no incident radiation on the boundaries of the slab. The medium simulates the expanding emission nebula.

The velocity law is given by:

$$V(\tau) = V_0 + \tau V_1,$$

with three values for the velocity gradient: $V_1 = 0$, $V_1 = -0.1$ and $V_1 = -0.2$. The parameters describing the line formation in the slab of the total optical thickness T are as follows: $\varepsilon = 10^{-3}$, $B = 1$, $T = 20$, $\delta = 1$. There is no continuum radiation.

We solved the above problem by means of the Jacobi and FBILI method. As the criterion where to stop the iterations we used the condition that the maximum relative change of the source function between two successive iterations $i-1$ and i

$$R_{\mathrm{c}}^{i} = |\frac{S^{i} - S^{i-1}}{S^{i-1}}|_{\max} \tag{56}$$

at all depth points is less than 10^{-3}.

With the Jacobi method the above condition was fulfilled in 64 iterations ($R_{\mathrm{c}} = 10^{-2}$ in 48). The normally emergent intensities (at $\mu = 1$) for three values of velocity gradient are shown in Fig. 2.

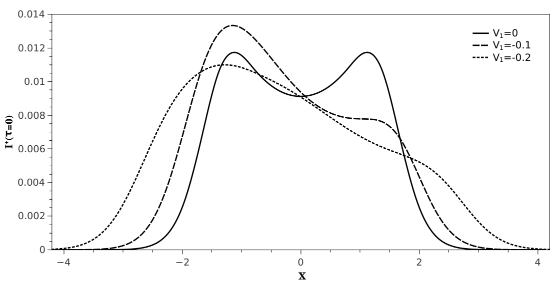

Fig. 2. *Emergent intensity at $\mu = 1$ from an expanding slab ($\varepsilon = 10^{-3}$, $B = 1$, $T = 20$) for three velocity gradients.*

When we applied the FBILI method with no iteration factor (Eq. (46)), the convergence was achieved in 35 iterations (i.e. in 27 iterations for the criterion $R_{\mathrm{c}} = 10^{-2}$). The use of iteration factor led to an almost exact solution in the first iteration but, afterwards, the procedure became unstable and failed to converge. As the use of the iteration factor improved the input solution a lot, we applied the iteration factor, i.e. Eq. (49) only in the first iteration and Eq. (46) in all the subsequent iterations. The convergence was stable and the criterion $R_{\mathrm{c}} = 10^{-3}$ was satisfied in only 24 iterations (i.e. $R_{\mathrm{c}} = 10^{-2}$ in 16 iterations).

The variation of the maximum relative change R_{c} with the iteration number for the Jacobi and the FBILI method is shown in Fig. 3.

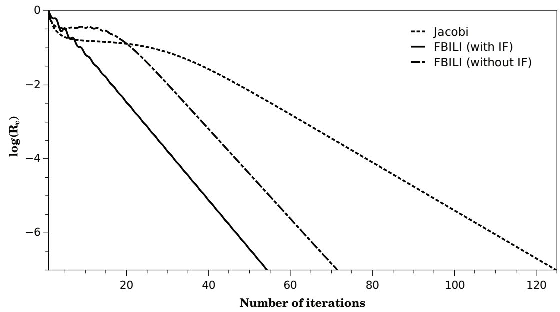

Fig. 3. *Maximum relative change R_{c} of the source function between two iterations as a function of iteration number for the Jacobi and FBILI method (with and without iteration factor) for the case of line formation in a plane-parallel expanding slab.*

4.2. Line formation in a spherical atmosphere

The use of the FBILI method for the solution of the line transfer in a spherically symmetric atmosphere (both static and expanding) is tested on benchmark problems proposed by Avrett and Loeser (1984).

We consider a stellar atmosphere consisting of homogeneous spherical shells. We take that the radius of the first layer $r_1 = 30$ and the last one $r_n = 1$ (in the units of stellar radius). Radial optical depth scale is defined as

$$\tau(r, x, \mu) = \int_{r}^{r(L)} \chi(r', x, \mu) dr',$$

where $\chi(r, x, \mu)$ is the total opacity (see Eq. (6)). The continuum and line opacities are given as $\chi^{\mathrm{C}}(r) = C_1/r^2$ and $\chi^{\mathrm{L}}(r) = C_2/r^2$, respectively. As the total continuum optical depth is 4, and the mean line optical depth is 1000, one finally has:

$$\chi(r, x, \mu) = \left[\frac{120}{29} + \frac{30000}{29}\phi(r, x, \mu)\right]\frac{1}{r^2}, \tag{57}$$

where the line profile ϕ is given by the Gaussian profile function (8). We used 24 Gauss-Legendre frequency points in the interval (-4,4).

Since the outward peaking of the radiation field in extended atmospheres necessitates a very fine angular mesh, we used 71 depth points (10 points per decade) and 72 rays (71 rays tangent to the layers and one more ray passing through the disk).

First, we had to get the solution for the background continuum source function. As proposed by Avrett and Loeser (1984), the continuum source function S^c is assumed to be of the form:

$$S^{\rm C} = \alpha J + (1-\alpha)B \ , \tag{58}$$

where the scattering coefficient is $\alpha = 0.5$ and the Planck function $B = 1$. With the opacity law given by $\chi^{\rm C}(r) = C_1/r^2$ and from the requirement that the total radial optical thickness of the atmosphere is 4, the radial continuum optical depth is given by

$$\tau(r) = \frac{120}{29}\left(\frac{1}{r} - \frac{1}{30}\right) .$$

We solved the monochromatic scattering case in the static spherical atmosphere using the FBILI (see Atanacković-Vukmanović 2003).

The above problem has been solved by many other authors: Mihalas et al. (1975), Rogers (1981), Avrett and Loeser (1984), Harper (1994), Gros et al. (1997). In Table 1 we list the results obtained by the FBILI method (our results coincide with those obtained by Atanacković-Vukmanović 2003) and the maximum relative differences between our solution and the other five. The results of Mihalas et al. (1975) are reported by Avrett and Loeser (1984). In the majority of cases the differences are about 1%. The maximum difference is 4% (at only one depth point) between our solution and that of the MKH. They can be due to different depth discretization and different approximations used to describe the behavior of $S(\tau)$ between any two consecutive depth points. All our computations were performed using the logarithmic spacing in the radial optical depth with 10 points per decade.

With the frequency-independent continuum source function (58) and the values of J given in Table 1, we solved the next test problem: line transfer with frequency independent background source function in a static spherical atmosphere. In this test problem the values $\varepsilon = 2 \cdot 10^{-3}$, $B = 1$, $\delta = 1$ are given. The results for the line source function $S^{\rm L}$ and the normalized flux profile F_x for this case are presented in Tables 2 and 3. In Table 2 we display our solution for the line source function $S^{\rm L}$ and maximum relative differences with respect to solutions obtained by other three authors. The line source function is in a very good agreement with all the three other methods. As we can see the maximum relative difference practically never exceeds 1%. These four different solutions for the line flux are shown in Table 3 together with the corresponding maximum relative differences. The normalized line profile is very similar to that of Harper and Rogers but is greater than that of Avrett and Loeser by less than 3%.

The convergence properties of the FBILI and Jacobi method for this specific problem are presented in Fig. 4. The maximum relative change $R_{\rm c}$ lesser than 10^{-2} and 10^{-3} is attained in 34 and 58 iterations, respectively by the Jacobi method, and in 16 and 29 iterations, respectively with the FBILI method.

Table 1. The mean monochromatic intensity J at selected radial distances obtained by the FBILI method (this work - PA) and the maximum relative differences $\Delta J[\%] = |\frac{J-J(PA)}{J(PA)}| \cdot 100\%$ with respect to the values obtained by other authors (Mihalas et al. (MKH), Rogers (R), Avrett and Loeser (AL), Harper (H), Gros et al. (GCS)).

r	J (PA)	ΔJ (MKH)	ΔJ (R)	ΔJ (AL)	ΔJ (H)	ΔJ (GCS)
30	0.0639	0.469	0.156	0.469	1.095	0.313
29.6	0.0675	0.593	0.148	0.148	0.897	0.000
29	0.0720	0.556	0.556	0.139	0.833	0.278
28	0.0789	0.634	0.507	0.253	0.887	0.253
26	0.0927	0.755	0.647	0.108	0.970	0.539
24	0.107	0.934	0.000	0.934	0.934	0.934
20	0.142	0.704	1.408	0.000	1.408	1.408
16	0.189	3.703	1.058	0.529	1.587	1.058
12	0.259	0.386	1.544	0.772	1.544	1.544
9	0.341	1.173	1.466	0.293	1.466	1.466
6	0.474	4.008	0.637	0.422	1.266	1.055
4	0.622	2.572	1.125	0.804	0.964	0.804
2	0.853	1.290	0.469	0.703	0.352	0.234
1	0.976	1.230	1.537	1.024	0.615	0.820

Table 2. The line source function S^{L} at selected radial distances of a static spherical atmosphere with frequency independent continuum source function obtained by the FBILI method (this work (PA)) and the maximum relative differences $\Delta S^{\mathrm{L}}[\%] = |\frac{S^{\mathrm{L}} - S^{\mathrm{L}}(PA)}{S^{\mathrm{L}}(PA)}| \cdot 100\%$ with respect to the solutions obtained by other authors (Mihalas et al. 1975 (MKH), Harper 1994 (H) and Rogers 1984 (R)).

r	S^{L} (PA)	ΔS^{L} (MKH)	ΔS^{L} (H)	ΔS^{L} (R)
30	0.04252	0.071	0.564	0.259
29.867	0.04944	0.384	0.243	0.425
28.365	0.09700	0.010	0.515	0.268
24.47	0.1928	0.156	0.882	0.311
18.456	0.3338	0.060	1.108	0.210
12.723	0.4840	0.186	1.198	0.413
8.93	0.6010	0.666	1.015	0.799
4.33	0.7872	0.877	0.699	0.368
2.0191	0.9271	0.539	0.464	0.076
1	0.9883	0.577	1.093	0.536

Table 3. The normalized flux profile $F(x)$ in the static case with a frequency independent continuum source function obtained by different numerical methods, and the corresponding maximum relative differences.

x	F (R)	F (AL)	F (H)	F (PA)	ΔF (R)	ΔF (AL)	ΔF (H)
0.0	0.4009	0.392	0.4037	0.4000	0.23	2.00	0.93
0.4	0.4237	0.414	0.4267	0.4244	0.16	2.45	0.54
0.8	0.5078	0.496	0.5112	0.5098	0.39	2.70	0.27
1.2	0.7024	0.685	0.7058	0.7002	0.31	2.60	0.80
1.6	1.0210	0.992	1.0231	1.0191	0.19	2.66	0.39
2.0	1.1835	1.154	1.1835	1.1787	0.41	2.09	0.41
2.4	1.0745	1.064	1.0743	1.0750	0.05	1.02	0.07
2.8	1.0113	1.010	1.0112	1.0102	0.11	0.02	0.10
3.2	1.0009	1.001	1.0010	1.0005	0.04	0.05	0.05
3.6	1.0000	1.000	1.0000	1.0000	0.00	0.00	0.00

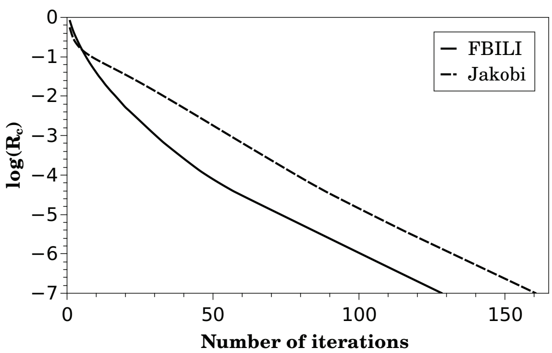

Fig. 4. *Maximum relative change R_{c} of the line source function between two iterations as a function of the iteration number for the Jacobi and FBILI method for the case of a line formation in a static spherical atmosphere.*

Finally, we solved the third benchmark problem of the line formation with frequency independent continuum source function in a spherical atmosphere ($\varepsilon = 2 \cdot 10^{-3}$, $B = 1$, $\delta = 1$) expanding according to the following velocity law:

$$V(r) = \frac{6}{\pi}\left[\arctan\left(\frac{2r-31}{29}\right) + \frac{\pi}{4}\right] , \qquad (59)$$

so that $V = 0$ at the bottom and $V = 3$ at the surface. Here, we used 40 Gauss-Legendre frequency points in the interval (-6,6).

The values of the line source function S^{L} in an expanding atmosphere and the line flux F_x for this case are given in Tables 4 and 5, respectively.

Our results agree with those of Harper to within 0.7% at all radial distances. The differences are again larger when compared with the solution obtained by the MKH (up to 3%).

Table 4. The line source function S^{L} at selected radial distances for the case of an atmosphere expanding according to the velocity law (59) obtained in this work (PA) and by other three authors, and the corresponding maximum relative differences between the solutions.

r	S^{L} (MKH)	S^{L} (H)	S^{L} (PA)	ΔS^{L} (MKH)	ΔS^{L} (H)
30	0.03652	0.03640	0.03646	0.165	0.165
29.99	0.03713	0.03717	0.03710	0.081	0.189
29.87	0.04264	0.04245	0.04233	0.732	0.283
29.29	0.05989	0.05980	0.05989	0.000	0.150
28.37	0.08194	0.08150	0.08201	0.085	0.622
26.87	0.1126	0.1119	0.1120	0.536	0.089
24.47	0.1573	0.1556	0.1568	0.319	0.765
21.53	0.2114	0.2085	0.2095	0.907	0.477
18.46	0.2737	0.2710	0.2692	1.672	0.669
15.86	0.3362	0.3287	0.3286	2.313	0.030
12.72	0.4293	0.4185	0.4165	3.073	0.480
8.93	0.5730	0.5572	0.5573	2.817	0.018
4.33	0.7903	0.7783	0.7805	1.256	0.281
2.02	0.9320	0.9248	0.9265	0.594	0.183
1	0.9827	0.9916	0.9883	0.567	0.334

Table 5. The normalized flux profile $F(x)$ for an atmosphere expanding according to the velocity law (59), obtained by Avrett and Loeser (AL), Harper (H) and in this work (PA), and the corresponding maximum relative differences between the solutions.

(a) red half

x	F (AL)	F (H)	F (PA)	ΔF (AL)	ΔF (H)
0.0	0.91	0.99	0.97	6.19	2.06
0.4	1.08	1.18	1.14	5.26	3.51
0.8	1.23	1.34	1.30	5.38	3.08
1.2	1.36	1.47	1.45	6.21	1.38
1.6	1.43	1.53	1.55	7.74	1.29
2.0	1.43	1.53	1.55	7.74	1.29
2.4	1.37	1.46	1.45	5.52	0.69
2.8	1.27	1.34	1.32	3.79	1.52
3.2	1.17	1.22	1.20	2.50	1.67
3.6	1.08	1.11	1.10	1.82	0.91
4.0	1.02	1.04	1.04	1.92	0.00
4.5	1.01	1.01	1.01	0.00	0.00
5.5	1.00	1.00	1.00	0.00	0.00

(b) blue half

x	F (AL)	F (H)	F (PA)	ΔF (AL)	ΔF (H)
-0.4	0.74	0.81	0.79	6.33	2.53
-0.8	0.59	0.64	0.63	6.35	1.59
-1.2	0.47	0.51	0.50	6.00	2.00
-1.6	0.39	0.43	0.42	7.14	2.38
-2.0	0.35	0.37	0.37	5.41	0.00
-2.4	0.34	0.37	0.36	5.56	2.78
-2.8	0.37	0.39	0.38	2.63	2.63
-3.2	0.43	0.45	0.44	2.27	2.27
-3.6	0.54	0.56	0.56	3.57	0.00
-4.0	0.72	0.74	0.74	2.70	0.00
-4.5	0.93	0.93	0.93	0.00	0.00
-5.5	1.00	1.00	1.00	0.00	0.00

The emergent normalized fluxes for the expanding semi-infinite atmosphere, obtained by the use of three different methods are also presented in Fig. 5.

The maximum relative change of the line source function with iterations for the moving case is shown in Fig. 6. We see that the differences in convergence properties between the Jacobi and FBILI methods in the moving case are less pronounced. The convergence criteria $R_\mathrm{c} = 10^{-2}$ and $R_\mathrm{c} = 10^{-3}$ are satisfied in 26 and 42 iterations with the Jacobi method while the same two criteria are attained in 16 and 29 iterations by the FBILI method.

Let us note here that the CPU time needed per iteration for both the Jacobi and FBILI method is practically the same. The only additional cost in the FBILI method is the computation of iteration factor (only one division) during the forward ("ingoing") step. The difference between the Jacobi and FBILI during the "outgoing" step is in the update of the source function: the FBILI does it *during* the backward step while Jacobi - *after*. This does not require any additional CPU time.

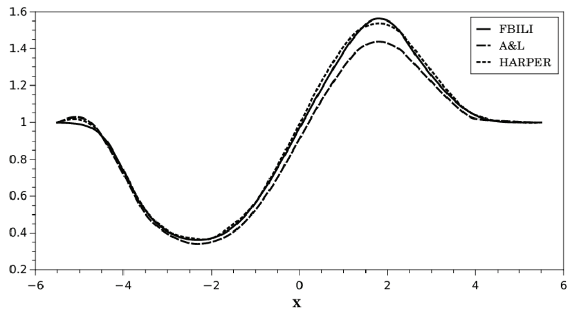

Fig. 5. *Line profile from an expanding atmosphere with a frequency independent continuum source function.*

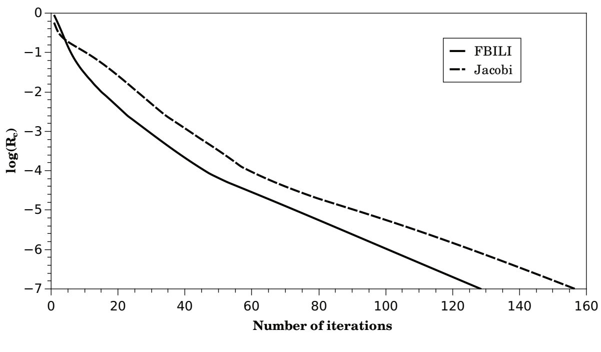

Fig. 6. *Maximum relative change R_c of the line source function between two iterations as a function of the iteration number for the Jacobi and FBILI method for the case of line formation in an atmosphere expanding according to the velocity law (59).*

We also computed the maximum relative true error of our solutions, i.e. the maximum relative difference of our solution with respect to the solution of a reference method:

$$T_\mathrm{e}^i = |\frac{S^i - S_\mathrm{MREF}^\infty}{S_\mathrm{REF}^\infty}|_\mathrm{max} \,. \qquad (60)$$

The true accuracy of the FBILI method is expressed with respect to S_REF^∞ - the fully converged "exact" solution obtained with some other well-tested RT code in a finer spatial grid. Here, we used the Jacobi solution as the reference (REF) one, in the first place because we could run the both methods (FBILI and Jacobi) straightforwardly getting the solutions for different choices of input parameters. For the "exact" solution S_REF^∞ we used the result obtained by the Jacobi method after the condition $R_\mathrm{c} = 10^{-12}$ is satisfied, with 20 depth points per decade.

The behavior of the maximum relative true error T_e with iterations is displayed in Fig. 7. We see that the so-called truncation error ($T_\mathrm{e}(\infty)$) as a measure of the true accuracy is about 1%. A suitable grid refinements can reduce the truncation error.

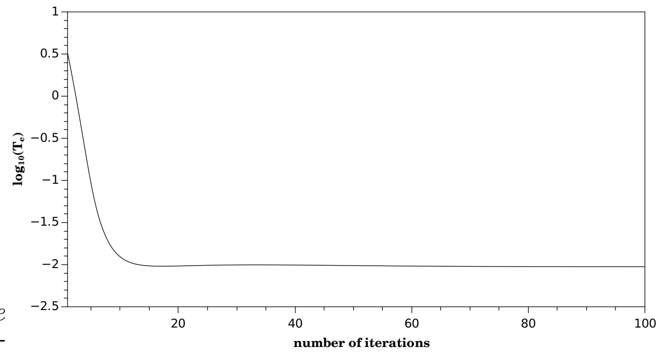

Fig. 7. *Maximum relative true error T_e as a function of the number of iterations in the solution of the line formation in an expanding atmosphere.*

5. CONCLUSIONS

In this paper we generalized the FBILI method to the RT problems in plane-parallel and spherically symmetric moving media with low velocity fields. The method is tested on the benchmark problems of the line formation in: plan-parallel expanding slab and expanding semi-infinite spherical atmosphere. The obtained solutions are accurate, i.e. in a good agreement with the results from four independent investigations performed by other authors. Due to the lack of published results on the convergence properties of the iterative procedures used by other authors to solve these benchmark problems, we compared the convergence rate of the FBILI method with that of the Jacobi one, which uses the same formal solver. When applied to a plane-parallel moving slab, the FBILI method is about 2.5-3 times faster. On the other hand, for spherically symmetric expanding atmosphere, the FBILI method is 1.4-1.8 times faster than the Jacobi method, which is a bit less than in the static case, where the convergence is 1.7-2 times faster. Acceleration by a factor of 2 and more provided by the FBILI method with no additional acceleration techniques, can be useful in moving stellar atmosphere calculations.

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FBILI МЕТОД ЗА РЕШАВАЊЕ ПРОБЛЕМА ФОРМИРАЊА ЛИНИЈА АТОМИМА СА ДВА НИВОА У ПОКРЕТНИМ СРЕДИНАМА МАЛИХ БРЗИНА

I. Pirković[1] and O. Atanacković[2]

[1]*Astronomical Observatory, Volgina 7, 11060 Belgrade 38, Serbia*

E–mail: *irena@aob.rs*

[2]*Department of Astronomy, Faculty of Mathematics, University of Belgrade Studentski trg 16, 11000 Belgrade, Serbia*

E–mail: *olga@math.rs*

УДК 52–645

Оригинални научни рад

У овом раду је уопштена брзо-конвергентна метода Двосмерно Имплицитне Ламбда Итерације (FBILI) на решавање проблема преноса зрачења у спектралној линији за атоме са два нивоа у покретним срединама малих брзина и коришћењем система посматрача. У циљу тестирања тачности и конвергентних својстава методе решили смо неколико астрофизички значајних тест-проблема формирања линија у условима не-ЛТР: у план-паралелној средини коначне дебљине у ширењу и у сферно-симетричним звезданим атмосферама, статичним и покретним. Упоредили смо добијена решења са решењима других аутора који су користили друге нумеричке методе.