Serb. Astron. J. № 190 (2015), 25 - 31 **UDC 524.834**
DOI: 10.2298/SAJ1590025M Original scientific paper

ON THE ε COSMOLOGICAL PARAMETER

Ž. Mijajlović[1], N. Pejović[2] and V. Marić[3]

[1]*State University of Novi Pazar, Department of Mathematics, ul. Vuka Karadžića bb, 36300 Novi Pazar, Serbia*
E–mail: *zarkom@matf.bg.ac.rs*

[2]*Department of Astronomy, Faculty of Mathematics, University of Belgrade Studentski trg 16, 11000 Belgrade, Serbia*
E–mail: *nada@matf.bg.ac.rs*

[3]*Serbian Academy of Sciences and Arts, Nikole Pašića 6, 21000 Novi Sad, Serbia*
E–mail: *vojam@uns.ac.rs*

(Received: March 22, 2015; Accepted: May 4, 2015)

SUMMARY: In the paper *On asymptotic solutions of Friedmann equations* **(Mijajlović et al. 2012), the theory of regularly varying functions in the sense of Karamata is applied in an asymptotic analysis of solutions of Friedmann equations. As is well known, solutions of these equations are used to represent cosmological parameters. Therefore, according to the theory of regularly varying functions all cosmological parameters depend on a function $\varepsilon(t)$ such that $\lim_{t\to\infty}\varepsilon(t) = 0$ and which appears in their integral representation. In this paper we derive a differential equation for the parameter $\varepsilon(t)$, discuss its solutions and give some physical interpretations.**

Key words. cosmological parameters, large-scale structure of Universe

1. INTRODUCTION

By cosmological parameters are usually meant some global physical quantities linked to the Universe. A good review of such approach for the ΛCDM model can be found, for example, in Lahav and Liddle (2014). Here we shall adopt a somewhat formalistic definition of cosmological parameters. By them we shall mean first of all solutions of Fiedmann equations, then any functions derived from these solutions and parameters by which the basic cosmological parameters can be expressed. While examples of the first two types of cosmological parameters are widely known, this is not the case for the third type. In our paper (Mijajlović et al. 2012) we exhibited such a parameter for the expanding universe with the cosmological constant Λ. This parameter is a continuous function $\varepsilon(t)$ having the limit 0 at infinity[1] and which appears in the Karamata representation of regularly varying functions (see Karamata 1930).

[1]Greek letters ε, ξ ζ, η and τ in this paper will be exclusively reserved for the names of continuous functions which have the limit 0 at infinity, e.g. $\lim_{t\to\infty}\varepsilon(t) = 0$. Therefore, any mention or use of a function with this name assumes the mentioned property. We shall often write shortly ε, η and so on instead of $\varepsilon(t)$, $\eta(t)$, etc.

The scale factor $a(t)$, the energy density $\rho(t)$, and material pressure in the universe $p(t)$ are usually taken as fundamental or basic cosmological parameters. These parameters are solutions of the Friedman equations (see Friedman 1924). We remind the reader that the Friedman equations are derived from the Einstein field equations. The Friedman equations are the following three ordinary differential equations:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2}, \qquad \text{Friedman equation,}$$
$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right), \qquad \text{Acceleration equation,}$$
$$\dot{\rho} + 3\frac{\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right) = 0, \qquad \text{Fluid equation.}$$

Let us mention that these three equations are not independent. For example, the fluid equation can be inferred from the other two equations. Therefore, for solving this system of essentially of two equations and three unknowns some additional condition is needed. Usually the equation of state $p = w\rho c^2$ is assumed. Here we shall discuss also some other conditions that are set to the unknowns $a(t)$, $p(t)$ and $\rho(t)$.

In our paper (Mijajlović et al. 2012), we applied the theory of regularly varying functions in asymptotic analysis of solutions of Friedman equations for cosmological parameters of the expanding universe. We proved that under certain but broad condition relating the acceleration equation, some of these parameters, including the acceleration parameter $a(t)$ and the Hubble parameter $H(t)$, are regularly varying functions. This is an integral limit condition which is a part of the Howard-Marić theorem, stated at the end of this section. One consequence is that, according to the representation theory for regularly varying functions, all these parameters depend on a function $\varepsilon(t)$ which is "hidden" in the integral representation of regularly varying functions. We shall see that the parameters $a(t)$, $\rho(t)$ and $H(t)$ uniformly depend only on $\varepsilon(t)$, while the parameters $p(t)$ and $q(t)$, besides $\varepsilon(t)$, depend explicitly on terms $\dot{\varepsilon}(t)$ and $\dot{\varepsilon}(t)t$ as well. While $\varepsilon(t) \to 0$ as $t \to \infty$, this is not necessary so for $\dot{\varepsilon}(t)$, or $\dot{\varepsilon}(t)t$ which may lead to various evolutions of cosmological parameters. Here we shall derive a differential equation for $\varepsilon(t)$ and discuss its possible solutions. This differential equation was announced in Mijajlović and Pejović (2015).

For better understanding we shall review briefly some basic concepts related to regular variation. Even though the modern theory of regular variations deals mainly with measurable functions, we shall assume here that all appearing functions are continuous in their domains and have a sufficient number of derivatives, at least a continuous second derivative. This assumption is clearly pursuant with the physical meaning of cosmological parameters. From the physical point of view, it means that events such as Big Crunch, or Big Rip are not included in our analysis. That is, in finite time t, $a(t) \neq 0$ and $a(t)$ does not become infinite. We are particularly concerned with the properties of regularly varying solutions of the second order differential equation

$$\ddot{y} + f(t)y = 0, \tag{1}$$

assuming that $f(t)$ is continuous on some time interval $[\alpha, \infty]$. Note that the acceleration equation has the form Eq. (1) if we take $y(t)$ for $a(t)$ and:

$$f(t) = \frac{4\pi G}{3}\left(\rho(t) + \frac{3p(t)}{c^2}\right). \tag{2}$$

This fact makes the acceleration equation central in our consideration. The reason is that the asymptotic analysis of solutions of Friedman equations reduces mainly to an analysis of solutions of the acceleration equation.

The notion of a regular variation is a form of a power law distribution. A weaker form of regular variation is described by the following power law relationship between quantities[2] F and t:

$$F(t) = t^r(\alpha + o(1)), \quad \alpha, r \in R. \tag{3}$$

Therefore, the simplest form of the power law is given by the equation $y = \alpha t^k$. Definition Eq. (3) of the power law can naturally be extended by use of the notion of a slowly varying function introduced by J. Karamata.

A real positive continuous function $L(t)$ defined for $x > x_0$ which satisfies:

$$\frac{L(\lambda t)}{L(t)} \to 1 \quad as \quad t \to \infty, \quad \lambda > 0, \tag{4}$$

is called a slowly varying function. A quantity F(t) is said to satisfy the generalized power law if:

$$F(t) = t^r L(t)\,, \tag{5}$$

where $L(t)$ is a slowly varying function and r is a real constant, so called the index of function $F(t)$. Hence, $F(t)$ is a regularly varying function if and only if $F(t)$ satisfies the generalized power law. Examples of slowly varying functions include $\ln(x)$ and iterated logarithmic functions $\ln(\ldots\ln(x)\ldots)$. In the rest of the paper the regularly varying functions will be abbreviated by RV functions, while the term "slowly varying" will be denoted by SV.

The following representation, according to Jovan Karamata, of RV and SV functions is of the great importance. It says that a function L is SV if and only if there are measurable[3] functions $h(x)$, a function $\varepsilon(t)$ and $b \in R$ so that:

$$L(x) = h(x)e^{\int_b^x \frac{\varepsilon(t)}{t}dt}, \quad x \geq b, \tag{6}$$

[2] In the following, R will denote the set of real numbers.

[3] For our purpose it is safe to assume $h(x)$ be continuous.

and $h(x) \to h_0$ as $x \to \infty$, h_0 is a positive constant. For further properties of RV functions, one may see Bigham et al. (1987).

Solutions L of differential equations that we are working with represent mechanical phenomena. Hence, as already mentioned, we may assume that L is a twice differentiable function. The function $\varepsilon(t)$ is not uniquely determined and due to the representation Eq. (6) where it is "covered" by the integral sign, we shall often call it a hidden parameter. Further, it is assumed that $L(t)$ is normalized, i.e. that $h(x)$ is a constant function. The class of normalized SV functions is denoted by $\mathcal{N}$. In Mijajlović et al. (2012) it was proved in a sequence of representation theorems that the fundamental cosmological parameters for the expanding universe depend essentially on the hidden parameter $\varepsilon(t)$. As the Friedman equations are invariant under translation transformation, this is also true for the expanding universe with the cosmological parameter Λ.

In our study of Friedman equations we used several results on RV solutions of the Eq. (1). There are various conditions for $f(t)$ that ensure that RV solutions of $\ddot{y} + f(t)y = 0$ to exist. We particularly used the following result, according to Howard and Marić (see Marić 2000) and (Kusano and Marić 2010):

Theorem *Let* $-\infty < \Gamma < 1/4$, *and let* $\alpha_1 < \alpha_2$ *be two roots of the equation:*

$$x^2 - x + \Gamma = 0. \tag{7}$$

Further, let L_i, *i=1,2 denote two normalized SV functions. Then there are two linearly independent RV solutions of* $\ddot{y} + f(t)y = 0$ *of the form:*

$$y_i(t) = t^{\alpha_i} L_i(t), \quad i = 1, 2, \tag{8}$$

if and only if $\lim_{x\to\infty} x \int_x^\infty f(t)dt = \Gamma$. *Moreover:*

$$L_2(t) \sim \frac{1}{(1 - 2\alpha_1)L_1(t)}.$$

The limit of the integral in the above theorem is central in our analysis and, in general, it is not easy to compute. However, it is easy to see that $\lim_{t\to\infty} t^2 f(t) = \Gamma$ implies $\lim_{x\to\infty} x \int_x^\infty f(t)dt = \Gamma$. Hence:

$$\lim_{t\to\infty} t^2 f(t) = \Gamma \tag{9}$$

gives a useful sufficient condition for the existence of RV solutions of the equation $\ddot{y} + f(t)y = 0$ as described in the previous theorem.

2. RV COSMOLOGICAL PARAMETERS

We already noted that the acceleration equation obviously has the form Eq. (1). Hence, under appropriate assumptions, we can apply the analysis of the previous section, in particular the Howard-Marić theorem. For this reason, we shall write from now on the acceleration equation in the form:

$$\ddot{a} + \frac{\mu(t)}{t^2} a = 0, \tag{10}$$

where:

$$\mu(t) = \frac{4\pi G}{3} t^2 \left(\rho(t) + \frac{3p(t)}{c^2} \right). \tag{11}$$

Then, the integral limit in the Howard-Marić theorem for the Eq. (10) looks like:

$$M(\mu) = \lim_{x\to\infty} x \int_x^\infty \frac{\mu(t)}{t^2} dt. \tag{12}$$

Functions for which this integral limit converges define the so called Marić class of functions $\mathcal{M}$. It is easy to see that M is a functional defined on $\mathcal{M}$. Also, according to the discussion following this theorem, we have:

$$\text{if} \quad \lim_{t\to\infty} \mu(t) = \Gamma \quad \text{then} \quad M(\mu) = \Gamma. \tag{13}$$

We note that the opposite of Eq. (13) does not hold. It is easy to find a function $\mu(t)$ such that $M(\mu)$ exists and is finite, but $\lim_{t\to\infty} \mu(t)$ does not exist[4].

In Mijajlović et al. (2012) Theorems 3.2 and 3.3, RV solutions of Friedman equations are found and in accordance with that, the cosmological parameters for Friedman non-oscillatory universe are determined. In fact, assuming that the integral limit $M(\mu)$ is convergent, say $M(\mu) = \Gamma$, the following is proven:

- if $\Gamma < 1/4$ then the universe is non-oscillatory.
- The converse is almost true, namely, if the universe is non-oscillatory then $\Gamma \leq 1/4$.
- If $\Gamma < 1/4$ and in some special cases for $\Gamma = 1/4$, the acceleration parameter $a(t)$, a solution of Friedman equations, is an RV function.

Assume that α is a root of the polynomial $x^2 - x + \Gamma$. Therefore:

$$\Gamma = \alpha(1 - \alpha). \tag{14}$$

Then the cosmological parameters are represented as follows:

scale factor $a(t)$: $a(t) = t^\alpha L(t)$, $\alpha \neq 0$ and L is an SV function. In other words, $a(t)$ is a regularly varying function of index α.

[4] Example from Mijajlović et al. (2012): $\mu(t) = \frac{1}{8} - 3t^2 \cos(t^3)$.

Hubble parameter $H(t)$:

$$H(t) = \frac{\alpha}{t} + \frac{\varepsilon}{t}. \tag{15}$$

Deceleration parameter $q(t)$:

$$q(t) = \frac{\mu(t)}{\alpha^2}(1+\eta) = \frac{1-\alpha}{\alpha} - \frac{t\dot{\varepsilon}}{\alpha^2}(1+\eta) + \tau. \tag{16}$$

Assuming the scale factor $a(t)$ satisfies the generalized power law we can introduce a new and useful constant w. It will appear that w is in fact the equation of state parameter. So, assume $a(t) = t^\alpha L(t)$, $L \in \mathcal{N}$ and $\alpha \neq 0$. We define w by:

$$w \equiv w_\alpha = \frac{2}{3\alpha} - 1. \tag{17}$$

Then the cosmological parameters can be put in a more standard form:

$$\begin{aligned} \alpha &= \frac{2}{3(1+w)}, & a(t) &= a_0 L(t) t^{\frac{2}{3(1+w)}} \\ H(t) &\sim \frac{2}{3(1+w)t}, & M(q) &= \frac{1+3w}{2} \end{aligned}. \tag{18}$$

Formulas for the exponent α and the Hubble parameter $H(t)$ are widely found in the literature. Formulas for $a(t)$ and $q(t)$ are also reduced to the standard formulas found in the literature if $L(t)$ and $q(t)$ are constant at infinity, or if the equation of state $p = w\rho c^2$ is assumed, or $\lim_{t\to\infty} t\dot{\varepsilon}(t) = 0$. We note that we did not assume any of these assumptions in derivation of Eq. (18). In fact, we found asymptotics for solutions of Friedman equations only by assuming $M(\mu) = \Gamma < 1/4$, and in certain cases for $\Gamma = 1/4$. As far as we know, it is widely assumed (implicitly) that the limit $\lim_{t\to\infty} \mu(t)$ exists and is finite, what is a much stronger assumption than that the integral limit $M(\mu)$ is convergent.

For the universe having flat curvature, one can infer the following weak form of the equation of state which gives a relation between pressure and density parameters.

There are functions ξ and ζ such that $p = \hat{w}\rho c^2$, where $\hat{w}(t) = w - t\dot{\xi} + \zeta$.

Therefore, if $t\dot{\xi} \to 0$ as $t \to \infty$, then $\hat{w}(t) \approx w$, which leads to $p = w\rho c^2$ the standard equation of state and classical asymptotics for cosmological parameters. In Mijajlović et al. (2012), Proposition 3.4, we found:

$$M(\mu) = \Gamma = \frac{2}{9} \cdot \frac{1+3w}{(1+w)^2}. \tag{19}$$

This allows us to give short derivations of asymptotics for cosmological parameters if the equation of state is assumed. We illustrate the method for the Hubble parameter $H(t)$. The acceleration equation is reduced to:

$$\ddot{a} + \frac{\Gamma}{t^2} a = 0, \tag{20}$$

and also we have the well known equation for the Hubble parameter:

$$\dot{H} + H^2 + \frac{\mu}{t^2} = 0. \tag{21}$$

By substitution $y(t) = H(-t)$ this equation is reduced to the Riccati equation:

$$\dot{y} = y^2 + \frac{\Gamma}{t^2}. \tag{22}$$

This equation has the general solution (see Polyanin and Zaitsev 2003 case 1.2.2-13)

$$y = \frac{\lambda}{t} - t^{2\lambda}\left(\frac{t^{2\lambda+1}}{2\lambda+1} + C\right)^{-1} \sim -\frac{\lambda+1}{t} \tag{23}$$

where λ is a root of $x^2 + x + \Gamma = 0$ and $t \to \infty$. Therefore:

$$H(t) = y(-t) \sim \frac{\lambda+1}{t} = \frac{\alpha}{t}, \tag{24}$$

where α is a root of $x^2 - x + \Gamma = 0$. Solving this quadratic equation in α for Γ given by Eq. (19) (or by Eqs. (17) and (19)) we immediately infer the formula for $H(t)$ in Eq. (18). In a similar manner, we may infer the asymptotics for other cosmological parameters. However, if the equation of state is not assumed, then for the solving of the Eq. (10) in general case for the non-oscillatory universe, i.e. with the condition $M(\mu)$ be convergent (and $M(\mu) < 1/4$), we need a much more subtle technique such as the theory of regular variations and Tauberian theory (see Korevar 2004), as shown in Mijajlović et al. (2012).

At this point we make a small historical digression. Professor John Barrow in a letter from October 2013 addressed our attention to his paper (Barrow 1998) where he established possible asymptotes for polynomial pressure and density behaviors. To find these asymptotics, Barrow applied some general theorems of Hardy and Fowler, which give the asymptotic behavior of all ultimately monotonic solutions of the first- and second-order polynomial differential equations, to the Einstein equations describing expanding universes. This theorem says that any rational function $H(x, y, y')$ is ultimately monotonic along a solution $y(x)$ of an algebraic differential equation of the form $f(x, y, y') \equiv \sum A x^m y^n y'^p = 0$. It is interesting that the asymptotics described by the Hardy theorem was also used by Chandrasekhar in the study of gravitational equilibrium of a gaseous configuration in stars in which the pressure and the density are related by a form of the equation of state (see Chandrasekhar 1938). The Hardy theorem comes from his theory *Orders of infinity* which preceded and influenced the Karamata theory of regular variation.

We see that the deceleration parameter $q(t)$, equation of state, and pressure $p(t)$ contain not only the "hidden" parameter $\varepsilon(t)$, but $\dot{\varepsilon}(t)$ and $t\dot{\varepsilon}(t)$ as well. While $\varepsilon(t) \to 0$ as $t \to \infty$, functions $\dot{\varepsilon}(t)$ and

$t\dot{\varepsilon}(t)$ may not have this property. In fact, they can be unbounded and oscillatory as well. This means that $q(t)$ and $p(t)$ can be unbounded and oscillatory as well. It seems that this fact is overlooked in classical cosmology, mainly due to the absence of "microscopic" analysis which gives us the theory of regularly varying functions. Therefore, it is of an interest to describe the hidden parameter ε in more details.

3. DIFFERENTIAL EQUATION FOR ε

Let $a(t)$ be a solution of the acceleration Eq. (10). Suppose that the integral limit $M(\mu)$ is convergent and that $M(\mu) = \Gamma < 1/4$. According to the previous section, these assumptions ensure that the acceleration parameter $a(t)$ is a normalized RV function. Finally, assume that ε is a "hidden" parameter in the representation Eq. (6) of $a(t)$. Under these conditions we shall prove that the function $\varepsilon(t)$ is a solution of the following general Riccati differential equation:

$$t\dot{\varepsilon} = (1-2\alpha)\varepsilon - \varepsilon^2 + \alpha(1-\alpha) - \mu(t), \qquad (25)$$

where α is a root of the polynomial $x^2 - x + \Gamma$. To prove this, first note that $a(t) = t^\alpha L(t)$ and

$$L(x) = c_0 e^{\int_b^x \frac{\varepsilon(t)}{t} dt}, \quad x \geq b,$$

for some constants c_0 and $b > 0$. Then we have $\dot{L} = \frac{\varepsilon}{t} L$, where from we find:

$$\ddot{L} = (\varepsilon^2 + \dot{\varepsilon}t - \varepsilon) L t^{-2}. \qquad (26)$$

After a short derivation directly from $a(t) = t^\alpha L(t)$ we find:

$$\ddot{a} = \ddot{L}t^\alpha + 2\alpha\dot{L}t^{\alpha-1} + \alpha(\alpha-1)Lt^{\alpha-2}. \qquad (27)$$

Substituting $\dot{L}$ and $\ddot{L}$ in this identity, we find

$$\ddot{a} = (\varepsilon^2 + \dot{\varepsilon}t - \varepsilon + 2\alpha\varepsilon + \alpha(\alpha-1))Lt^{\alpha-2}. \qquad (28)$$

Substituting so obtained $\ddot{a}$ and a in the acceleration Eq. (10) and canceling appropriate terms we obtain the differential equation Eq. (25).

We note that by Eq. (14) there is another form of the Eq. (25):

$$t\dot{\varepsilon} = (1-2\alpha)\varepsilon - \varepsilon^2 + \Gamma - \mu(t). \qquad (29)$$

We also remark the striking similarity of the Eq. (25) to Emden's differential equation, well known to play a fundamental role in the study of the internal structure of the stars, see for example Hopf (1931).

We continue now to discuss properties of solutions of the Eq. (25). Functions having the limit 0 at infinity are mapped by the functional M into 0. Hence $M(\varepsilon) = 0$ and $M(\varepsilon^2) = 0$. Also $M(\mu) = \Gamma$ and $M(\Gamma) = \Gamma$. We have now to distinguish between two cases:

a. There exists $\lim_{t\to\infty} \mu(t)$.
b. $\lim_{t\to\infty} \mu(t)$ does not exist but, of course, the integral limit $M(\mu)$ does exist.

In the former case, $t\dot{\varepsilon}(t)$ tends to 0 as $t \to \infty$. Hence, applying functional M on the Eq. (29), we obtain:

$$M(t\dot{\varepsilon}) = 0. \qquad (30)$$

Actually, this formula is valid for any continuously differentiable function which tends to zero as $t \to \infty$ and this can be proved by partial integration.

By remark Eq. (13), which in fact holds for arbitrary continuous functions, we infer the following result:

$$\textit{if } \lim_{t\to\infty} t\dot{\varepsilon}(t) \textit{ exists, then } \lim_{t\to\infty} t\dot{\varepsilon}(t) = 0. \qquad (31)$$

In particular, if $\lim_{t\to\infty} \mu(t) = \Gamma = \alpha(1-\alpha)$, then $t\dot{\varepsilon}$ has the limit 0 at infinity, hence $\dot{\varepsilon}$ and $t\dot{\varepsilon}$ can be neglected in representation of cosmological parameters. In this case, let us call it the tame case, $q(t)$ and the state equation reduce to their standard form in classical cosmology:

$$q(t) = \frac{1+3w}{2}, \quad p(t) = w\rho(t)c^2.$$

The later, non-tame and particularly interesting case is when $\lim_{t\to\infty} \mu(t)$ does not exist but anyhow $M(\mu) = \Gamma = \alpha(1-\alpha)$. Then the "hidden" parameter ε, in fact its derivative $\dot{\varepsilon}$, might have a strong influence in asymptotical behavior of parameters $q(t)$, $p(t)$ and the equation of state. In this case one can show, for example, that the function $\xi(t) = t\dot{\varepsilon}(t)$ oscillates infinitely many times, i.e. it intersects the time-axes infinitely many times. This property of ξ would induce variations of deceleration parameter $q(t)$ around the value:

$$\frac{1-\alpha}{\alpha} = \frac{1+3w}{2} \qquad (32)$$

see Eqs. (16) and (18), and also of the energy pressure $p(t)$, see the weak form of the equation of state.

If $\zeta = (1-2\alpha)\varepsilon - \varepsilon^2$, then Eq. (29) may be written as:

$$t\dot{\varepsilon} = \zeta + \Gamma - \mu(t)\,. \qquad (33)$$

The function $\zeta(t)$ has the limit 0 at infinity, wherefrom we have the following relation:

$$t\dot{\varepsilon} \sim \Gamma - \mu(t)\,, \qquad (34)$$

which is particularly interesting when $\lim_{t\to\infty} t\dot{\varepsilon}(t)$ does not exist. From the relation Eq. (34) we see once again that if we assume $t \to \infty$ then $t\dot{\varepsilon}(t) \to 0$ if and only if $\mu(t) \to \Gamma$.

We also observe that under the substitution $\eta = (\varepsilon + \alpha)/t$ the Eq. (25) is reduced to:

$$\dot{\eta} + \eta^2 + \frac{\mu}{t^2} = 0. \qquad (35)$$

Note that this the same differential equation Eq. (21) whose solution is the Hubble parameter $H(t)$. Therefore, $H(t) \sim \eta(t)$, pursuant to the asymptotics of $H(t)$ we found in Eq. (24).

4. DISCUSSION

We shall discuss briefly some possible physical explanations or models assuming the non-tame case. The explanation is proposed taking into account properties of the "hidden" parameter $\varepsilon(t)$.

1. Dark matter and dark energy are in the equilibrium but fluctuations of this state produce variation of $\hat{w}(t)$, $q(t)$ and $p(t)$.

2. Variations of $q(t)$ and $p(t)$ are consequences of the extremely rapid expansion of the Universe which appeared in the inflationary epoch (about 10^{-36} seconds after the Big Bang). We may think of these variations as an echo effect due to thermalization which appeared when the inflation epoch ended (about 10^{-32} seconds after the Big Bang).

3. Variations of $q(t)$ and $p(t)$ are consequences of the existence and influence of the dual universe. Even though it may sound as a mathematical fiction, we can easily and explicitly find the dual set of "cosmological parameters" starting from the second fundamental solution $L_2(t)$ in the Howard - Marić theorem applied to the acceleration equation. To find the dual set of cosmological parameters we take the second root $\beta = 1 - \alpha$ of the quadratic equation $x^2 - x + \Gamma = 0$ appearing in this theorem. Now we use β instead of α for the index of the RV solution $a(t)$ - deceleration parameter and for determination of other constants and cosmological parameters. For example, we derived in [Mijajlović at al., 2012] the following symmetric identity for equation of state parameters:

$$w_\alpha + w_\beta + 3w_\alpha w_\beta = 1 \tag{36}$$

For our universe we have $w = w_\alpha$, while for the dual universe the corresponding equation of state parameter is w_β. In a similar manner one can deduce formulas for other parameters. If one wants to give any physical meaning to the so obtained dual set of functions, it is rather natural to interpret them as cosmological parameters of the dual universe.

We note that any of the proposed explanations does not exclude the validness of the other two. If these models, or some of them are valid, then the variations of cosmological parameters in the non-tame case may be seen also as the resultant of their mutual interference.

5. CONCLUSION

This paper is a continuation of the asymptotical analysis of solutions of Friedman equations that we started in our paper (Mijajlović et al. 2012), using the theory of regularly varying functions. Here we discussed the possible importance of properties of the ε parameter which appears in the representation of regularly varying functions. Clearly two possibilities are distinguished: the tame case described by $\lim_{t\to\infty} t\dot{\varepsilon}(t) = 0$, and the opposite, when this limit does not exist. We also found the differential equation that $\varepsilon(t)$ must satisfy and we discussed properties of solutions to this equation.

Acknowledgements – We are grateful to the Reviewer for a valuable information and suggestions concerning some early works on Raychaudhuri equation written in the form $d^2x/dt^2 + F(t)x = 0$. This equation has the same form as the equation which is the subject of the Howard-Marić theorem. The Reviewer's remarks are particularly related to Tipler's papers (Tipler 1977, 1979) where we found some theorems on asymptotic behavior of solutions to this equation. By his opinion, the method presented in our paper might be applied in this case as well. We remind the reader that the Raychaudhuri equation has an important role in description of the gravitational focusing and space-time singularities.

During the work on this paper the first two authors were financially supported by the Ministry of Education and Science of the Republic of Serbia through the project III 44006.

REFERENCES

Barrow, J. D.: 1996, *Class. Quantum Grav.*, **13**, 2965.
Barrow, J. D. and Shaw, D. J.: 2008, *Class. Quantum Grav.*, **25**, 08 5012.
Bingham, N. H., Goldie, C. M. and Teugels, J. L.: 1987, *Regular variation*, Cambridge Univ. Press, Cambridge, 491.
Chandrasekhar, C: 1938, An Introduction to the Study of Stellar Structure, Univ. of Chicago Press, Chicago, 509.
Friedman, A.: 1924, *Z. Phys.*, **21**(1), 326.
Hopf, E.: 1931, *Mon. Not. R. Astron. Soc.*, **91**, 653.
Karamata, J.: 1930, *Math. (Cluj)*.
Korevaar, J.: 2004, Tauberian theory, Springer, Berlin, 483.
Kusano, T. and Marić, V.: 2010, *Publ. Inst. Math. (Belgrade)*, **88**(102), 1.
Lahav, O. and Liddle, A. R.: 2014, arXiv:1401.1389v1.
Marić, V.: 2000, *Regular Variation and Differential Equations*, **127**, Springer, Berlin, 127.
Mijajlović, Ž., Pejović, N., Šegan, S. and Damljanović, G.: 2012, *Applied Mathematics and Computation*, **219**, 1273.
Mijajlović, Ž. and Pejović, N.: 2015, in Proc. XVII Conf. Astronomers of Serbia, in press.
Polyanin, A. D. and Zaitsev, V. F.: 2003, Handbook of exact solutions for ordinary differential equations, Chapman & Hall/CRC, Boca Raton, London, 803.
Tipler, F. J.: 1977, *Phys. Rev. D*, **15**, 942.
Tipler, F. J.: 1979, *Gen. Rel. Gravitation*, **10**, 1005.

О КОСМОЛОШКОМ ПАРАМЕТРУ ε

Ž. Mijajlović[1], N. Pejović[2] and V. Marić[3]

[1]*State University of Novi Pazar, Department of Mathematics, ul. Vuka Karadića bb, 36300 Novi Pazar, Serbia*

E–mail: *zarkom@matf.bg.ac.rs*

[2]*Department of Astronomy, Faculty of Mathematics, University of Belgrade Studentski trg 16, 11000 Belgrade, Serbia*

E–mail: *nada@matf.bg.ac.rs*

[3]*Serbian Academy of Sciences and Arts, Knez Mihailova 35, 11000 Belgrade, Serbia*

E–mail: *vojam@uns.ac.rs*

УДК 524.834

Оригинални научни рад

У раду *On asymptotic solutions of Friedmann equations* (Mijajlović et al. 2012), применили смо теорију регуларно промењљивих функција у асимптотској анализи решења Фридманових једначина. Као што је добро познато, решења ових једначина репрезентују космолошке параметре. Дакле, према теорији регуларно промењљивих функција, сви космолошки параметри зависе од функције $\varepsilon(t)$ која се појављује у њиховој интегралној репрезентацији и за коју важи $\lim_{t\to\infty} \varepsilon(t) = 0$. У овом раду извели смо диференцијалну једначину за $\varepsilon(t)$, дискутовали решења те једначине и дали неке физичке интерпретације.