Serb. Astron. J. № 195 (2017), 1 - 8
DOI: https://doi.org/10.2298/SAJ170407005K

UDC 523.44-323.8
Invited review

COMPUTATION OF ASTEROID PROPER ELEMENTS: RECENT ADVANCES

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(Received: April 7, 2017; Accepted: April 7, 2017)

SUMMARY: The recent advances in computation of asteroid proper elements are briefly reviewed. Although not representing real breakthroughs in computation and stability assessment of proper elements, these advances can still be considered as important improvements offering solutions to some practical problems encountered in the past. The problem of getting unrealistic values of perihelion frequency for very low eccentricity orbits is solved by computing frequencies using the frequency-modified Fourier transform. The synthetic resonant proper elements adjusted to a given secular resonance helped to prove the existence of Astraea asteroid family. The preliminary assessment of stability with time of proper elements computed by means of the analytical theory provides a good indication of their poorer performance with respect to their synthetic counterparts, and advocates in favor of ceasing their regular maintenance; the final decision should, however, be taken on the basis of more comprehensive and reliable direct estimate of their individual and sample average deviations from constancy.

Key words. Celestial mechanics – Minor planets, asteroids

1. INTRODUCTION

Asteroid proper orbital elements are the approximate quasi-integrals of the full N-body equations of motion of these bodies. Alternatively, they can represent the true integrals, but of the simplified dynamics. As such they are nearly constant in time, thus preserving the dynamical signature of the events that took place in the remote past. Proper elements are used as parameters for classification of asteroids into families, and for studies of asteroid long-term dynamics.

The computation of asteroid proper elements is based on averaging over fast and slow angles, taking advantage of degeneracy of the dynamical system (the solar system with the Sun as a central attractor and major planets as perturbing bodies). By averaging the instantaneous osculating orbital elements, inferred from observations, over the fast angles (e.g. mean longitudes of both, asteroids and perturbing planets), the short-periodic perturbations are accounted for and eliminated from the initial conditions, providing the so-called mean elements. The averaging of mean elements over slow angles (longitudes of perihelia and nodes of asteroids and planets) then gives the proper elements. In a way, consequently, the proper elements are representative of an "average" motion of asteroids over very long time scales.

The proper elements that are typically used as parameters for classification of asteroids into families are three: the proper semi-major axis a_p, proper eccentricity e_p, and sine of proper inclination $\sin I_p$, while for the study of dynamical evolution of asteroids over very long time spans – hundreds of millions to billions of years – the proper longitude of perihelion ϖ_p, and proper longitude of node Ω_p, and/or their precession rates g and s, are used as well.

Asteroid families originate either in highly energetic collisions among the asteroids in which the parent bodies are completely fragmented, or in somewhat less energetic cratering events when the typical outcome consists of the impact crater on the large asteroid and a swarm of comparatively small fragments forming the family. In either case, the fragments originating in collision acquire relative velocities which are small in comparison to their orbital velocities, thus remaining close to each other and to the parent body in the phase space of proper orbital elements. In turn, they are therefore recognizable in this space as density contrasts with respect to the background asteroids, even if formed a long time ago.

Families of asteroids are discovered as early as in 1918 (Hirayama 1918), when only several hundred asteroids were known and only a handful of families was identified. The most advanced contemporary classifications (Masiero et al. 2013, Milani et al. 2014, Nesvorný et al. 2015) make use of several hundred thousand asteroids and contain well over 100 statistically significant families.

In a recent review (Knežević 2016), the history of the discovery of families and the current status-of-the-art in the field are described in full detail. A special attention has been paid to advances achieved over the past century in computation of parameters for the family classification, that is to the development of more and more sophisticated and accurate theories, resulting in proper elements of significantly improved stability and valid for longer and longer time spans. In this paper, we extend this description to some more recent advances, including small modifications of the existing theories to cope with some specific problems, but also the more substantial improvements in computation of proper elements for especially demanding resonant dynamics, and of the stability estimates for analytically computed proper elements.

2. COMPUTATION OF ASTEROID PROPER ELEMENTS

Very soon after the discovery of families, Hirayama (1922) recognized that some kind of invariable elements should be used instead of the ordinary osculating elements in order to detect the families. Such elements have been found in the form of amplitudes of free oscillations in solutions of the Lagrange-Laplace linear theory of secular perturbations. These amplitudes are called *proper elements*, and their determination for even a modest number of asteroids known at the time was enough to recognize several most prominent asteroid families (Themis, Eos, Koronis, Maria), identified also in all later classifications up to the present. Subsequent developments regarding the computation of proper elements of improved accuracy were due first to an upgrade of the linear theory (Brouwer 1951), and then to the development of a new semi-analytical theory (Williams 1969). The proper elements computed using these theories were presumably more stable than the previously available ones, thus enabling more reliable identification of the members of already known families, as well as the discovery of many new ones (due also to the increasing number of known asteroids). In a while, however, these proper elements proved still to be of unsatisfactory quality, even lacking the error estimates, while the families, in particular smaller ones, identified by using these elements, often exhibited heterogeneous composition of members incompatible with common origin (Carusi and Valsecchi 1982, Zappalà et al. 1984).

The situation changed for the better when the new theories to compute proper elements, as well as new methods of classification were developed and applied to a larger asteroid sample (Zappalà et al. 1990). Regarding the computation of proper elements, the decisive breakthrough came with the new analytical theory of asteroid secular perturbations (Milani and Knežević 1990) accounting for the terms in expansion of the perturbing function of up to degree four in eccentricity and inclination, in the first order with respect to perturbing mass, and of degree two in the second order. Based on the Lie series canonical transformations of perturbing potential and twofold averaging to compute first the mean and then the proper elements, this theory provided results of unprecedented accuracy, enough for the purpose of a reliable family classification. Later upgrades of the theory (Milani and Knežević 1992, 1994), as well as a dedicated study (Knežević et al. 1991), revealed the existence and importance of the so-called non-linear secular resonances, pinpointed locations of the most important linear and non-linear secular resonances in the asteroid Main Belt and in the solar system as a whole, and enabled analyzes of their interactions with families.

The analytical theory proved to be very efficient in terms of the computation of proper elements, but, being based on the truncated expansion of the perturbing potential, applicable only to asteroids of low to moderate eccentricities and inclinations. This drawback, together with a fast increase of the number of known asteroids, which in itself argued for even more accurate proper elements needed for reliable family classification in an enlarged asteroid sample with increasingly dense background, led to the development of the synthetic theory to compute proper elements (Knežević and Milani 2000, 2003). The synthetic theory consists of a set of purely numerical procedures to perform the same averaging over the fast and slow angles as in the analytical theory: the numerical integration of the asteroid orbits over millions of years carried out in the framework of a suitable dynamical model, on-line digital filtering of the short-periodic perturbations to produce mean elements, removal of the forced oscillations and the Fourier analysis of secular time series to remove long-periodic perturbations and compute the proper elements. The computation of proper elements by means of this theory is much more time consuming than with the analytical one, but can be done for asteroids of any eccentricity and inclination, and provides by a factor of at least 3 more accurate results. Hence, this theory is nowadays a standard one, used almost exclusively to produce the best proper elements for all known asteroids.

The current release of catalog of proper elements, available via the AstDyS service[1], contains data for more than half a million asteroids, the typical accuracy of which corresponds to a relative velocity of family members of 15 m/s, well below the typical escape velocity from family parent bodies.

In addition to the above theories intended for the mass production of proper elements for asteroids in the main belt, various specially adapted theories were developed for specific asteroid populations: the semi-analytical theories for secular resonant asteroids (Morbidelli 1993) and for high-inclination and high-eccentricity objects (Lemaitre and Morbidelli 1994), the theories for Trojans (Milani 1993, Beaugé and Roig 2001) and Hildas (Schubart 1991), etc.

3. RECENT ADVANCES

There were no major breakthroughs in the recent past, when computation of proper elements is in question, but some important advances in the field deserve to be mentioned here.

3.1. Fundamental frequencies for low proper eccentricity objects

One such advancement pertains to the computation of fundamental frequencies and their use as parameters for family classification (Carruba and Michtschenko 2007, 2009). The fundamental frequencies are the average rates of precession of the angular orbital elements: n is the mean daily motion, or the precession rate of the fast angle, e.g. mean longitude λ, while, as mentioned above, g, s are the precession rates of slow angles - the proper longitude of perihelion ϖ_p, and the proper longitude of node Ω_p, respectively. In the analytical theory they are computed from rather complicated expressions (Knežević et al. 1991), and suffer from large uncertainties in the vicinity of strong mean motion resonances. In the synthetic theory (Knežević and Milani 2000) they are computed by a linear fit to the time series of the corresponding proper angle.

The synthetic frequencies computed from the fit are generally of good accuracy except in a rare case of orbits with extremely low eccentricities and high inclinations, when the frequency g can acquire unrealistic values. As discussed in Knežević and Milani (2000), in such a case it is actually the computation of proper longitude of perihelion which fails, the angle being indeterminate and the resulting fit affected by large uncertainties.

Such a problem has been detected in the family of (480) Hansa, and subsequently analyzed by Carruba (2010, see Fig. 5a; also Fig. 1 below). More than 350 objects with proper eccentricities $e_p < 0.0179$ were found to exhibit erroneous, typically large negative, values of frequency g, and this was ascribed to the fact that the standard numerical procedure (Knežević and Milani 2000) fails if the free oscillations are smaller than the forced ones ("paradoxical librations"; see Milani and Nobili 1984, Beaugé and Roig 2001). As Carruba emphasizes, in these cases the angle ϖ_p does not describe the full circle in the corresponding polar coordinates $h = e_p \sin \varpi_p, k = e_p \cos \varpi_p$, but rather oscillates (librates) between the resonant boundaries. Interestingly, the three proper elements used as parameters for family classification appear to be unaffected by this problem.

The remedy to this problem has been proposed in Carruba (2010), and it consists in computation of frequencies by using the frequency-modified Fourier transform (FMFT; Šidlichovský and Nesvorný 1997) on time series of osculating equinoctial elements of integrated asteroid orbits. The following simple procedure has been used: first all planetary frequencies are eliminated from spectra of the Fourier transform of the equinoctial elements, and then the largest values in the spectra that are still observable are assigned as proper frequencies. As demonstrated by Fig. 5b in Carruba (2010), showing the results of the application of FMFT, there are no more anomalous values seen in Fig. 5a (and our Fig. 1), showing the results from the fit, and the resulting g frequencies are in full agreement with those computed by either method for the objects with proper eccentricities larger than the above critical value.

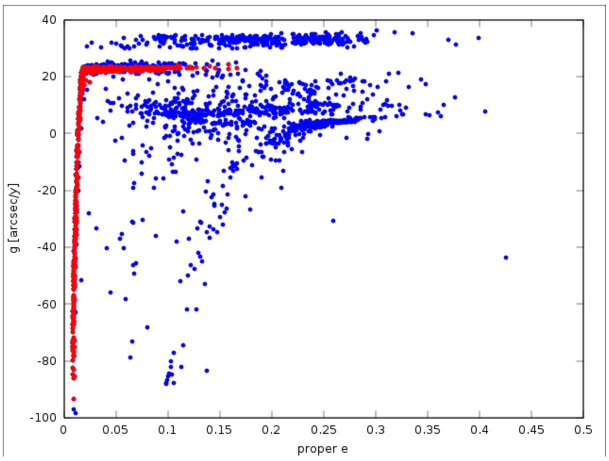

Fig. 1. *Frequencies g as a function of proper eccentricity. Members of the Hansa family are marked in red, and the other asteroids in the region in blue. Frequencies for asteroids with nearly circular orbits* ($e_p < 0.0179$) *exhibit a sharp drop due to the failure of the numerical procedure used to compute them.*

The question, however, to be addressed in the forthcoming analyses is whether in the proper elements catalog there are other cases of paradoxical libration affecting the families.

[1]http://hamilton.dm.unipi.it/astdys2

3.2. Asteroid synthetic resonant proper elements

The family of (5) Astraea has been recognized only recently by Milani et al. (2014) as a large concentration of asteroids locked in the nonlinear secular resonance $g + g_5 - 2g_6$, where g_5 and g_6 are the frequencies of the perihelia of Jupiter and Saturn, respectively. As a consequence, the synthetic proper elements of the family members, as well as of the other, background asteroids locked in the same resonance, suffered in many cases from large instabilities because the resonant oscillations could not have been completely removed. On top of that, the averaging of perturbations for resonant orbits fails since the critical argument librates, thus producing a well known effect of alignment of proper elements, proper eccentricity in this case, along the libration center. In turn, this apparent compression in combination with a large spread of the family in proper eccentricity produces a strangely elongated shape of the family in the proper semimajor axis – proper eccentricity projection, which is difficult to interpret in terms of a plausible collisional model.

All this raised doubts regarding the existence of the family, some researchers even claiming that the observed density contrast is just an artifact of the erroneously computed and compressed proper elements. For the existence of the family there were, on the other hand, some independent arguments, like the existence of the so-called V-shape dependence of the asteroid absolute magnitude (or inverse size) on the proper semimajor axis (see e.g. Vokrouhlický et al. 2006), due to the non-gravitational Yarkovsky effect (Rubincam 1995) and typical of families, then the spectral homogeneity of the family members expected for the bodies of common origin, etc.

As already mentioned, there exists a semi-analytical theory for secular resonant asteroids by Morbidelli (1993), but it has never been used to compute proper elements for the family classification purposes. The reasons are two: (i) these elements are adapted to a specific resonance and must be computed separately for each of them; (ii) their definition is different from the definition of proper elements for non-resonant asteroids, thus they cannot be used together to identify the families which cross the resonance, or spill over its boundaries.

In the case of the proposed family of (5) Astraea, however, neither of the two reasons apply: as explained above, it is entirely contained within a single nonlinear secular resonance, and it can be considered as essentially isolated from the background outside the resonance. For the background asteroids within the resonance, the same kind of resonant proper elements can be computed, in order to ensure the correct application of the classification procedure. This approach has been used in Milani et al. (2016) to resolve the issue, but by using a purely numerical procedure and thus producing the synthetic resonant proper elements.

Technically speaking, the algorithm to compute resonant proper elements, specifically adapted to the secular resonance $g + g_5 - 2g_6$, has been applied only to the resonant asteroids in the zone surrounding the asteroid (5) Astraea, namely those with librating critical argument $\varpi + \varpi_5 - 2\varpi_6$. As the resonance in question involves only perihelia, the proper inclination I_p is very little affected by this resonance, thus standard averaging over a long enough time is sufficient to get only small instabilities: $\sigma(\sin I_p) > 0.005$ occurs for only 1.2% of the resonant cases. The proper semimajor axis a_p is also unaffected and is computed by the usual synthetic method. A special algorithm was needed only for the proper eccentricity e_p, which actually had to be substituted with a more appropriate quantity, adjusted to the resonance in question. For this purpose, the amplitude Δe of libration in eccentricity, associated with the libration of the critical argument, was used as the most suitable resonant proper element (see Fig. 10 in Milani et al. 2016). The starting point for its computation was the output of a 10 Myr numerical integration, filtered on-line (Knežević and Milani 2000) to remove all oscillations with periods up to 300 yr, and sampled every 200 yr. Since the period of libration is very long (typically > 1 Myr) filtering is applied again with decimation 100, an output every 20 000 yr, and the oscillations with periods up to 30 000 yr removed. The next phase was looking to the doubly filtered time series of the eccentricity, and computing the spectrum for periods in the interval between 1 and 6 Myr. Then the maximum spectral density has been selected, and the corresponding amplitude used as proper element.

The improvement in the stability with time of proper amplitude of libration Δe_p, with respect to the one of the synthetic proper eccentricity e_p, was found to be very significant. A total of 8 972 sets of resonant proper elements in the Astraea zone have been computed in this way, including, for the moment, only numbered asteroids.

The result of applying the family classification procedure to the new set of resonant proper elements was remarkable indeed. Plots of the resonant proper elements of all asteroids locked in the resonance, in both, the $(a_p, \Delta e_p)$ and $(\sin I_p, \Delta e_p)$ planes show such a prominent and regularly shaped concentration (see Fig. 23 in Milani et al. 2016), with (5) Astraea close to the center, that it is difficult to doubt the presence of a very large family. Since the resonant region does not possess a regular box-like shape, the usual classification procedure had to be slightly modified, but the family of (5) Astraea with 5 192 members was unmistakably identified. The shape of this family in the $(a_p, \Delta e_p, \sin I_p)$ space is fairly regular, family does not completely fill the resonant region, actually the termination of the family in Δe_p, in a_p and in $\sin I_p$ has nothing to do with the boundary of the resonant orbits; thus, the assumption that the entire Astraea group is simply an island of asteroids isolated by the resonance can also be ruled out. It can be concluded that a family of (5) Astraea, originated in a large cratering event, does exist, and that the computation of special proper elements in this case enabled to definitely resolve the issue.

3.3. Instabilities of analytical proper elements

As already pointed out above, the analytical proper elements are easy to compute once the otherwise complicated and demanding theory is coded employing an efficient computer algebra system (Knežević 1992). These elements , however, suffer from two important drawbacks: their availability only for objects with orbits of low to moderate eccentricity and inclination, and lack of the precise estimate of their uncertainties. The former problem was not considered too important, because asteroids in general have orbits of rather low eccentricity and inclination, so that the analytical theory could have been applied to the vast majority of known asteroids, and because a special semi-analytical theory by Lemaitre and Morbidelli (1994) has been developed to handle the high eccentricity/inclination cases. For the latter, more serious problem, an attempt, with only a quite limited result, to address it, has been made by computing the accuracy estimate for a small set of asteroids representative of the families (Milani and Knežević 1990). Namely, the deviation from constancy of the time series of proper elements for a dozen asteroids, integrated numerically for 400 kyr (even for 1, 5 Myr in a couple of cases, which was at the limit of the performance of the then computers) and sampled every 200 yr, has been estimated by computing the proper elements for each sampled epoch over the whole period of integration. The obtained standard deviation was used as a measure of the accuracy. Simultaneously, the uncertainty of the fundamental frequencies has been estimated by comparison of the computed values with the slope of the fit to the time series of proper angles. The typical uncertainty was found to be on the order of $2-3\cdot10^{-3}$ in proper eccentricity, and $1-2\cdot10^{-3}$ in proper inclination, while the corresponding uncertainties of the proper semi-major axes were typically one or two orders of magnitude smaller.

The computation of analytical proper elements has been carried out for more than 25 years as a part of the regular maintenance of the catalog available via the AstDyS service. One of the reasons for such longevity of the analytical computations, performed even in parallel with the computation of synthetic proper elements of superior accuracy, was that they have been regularly determined also for the multi-opposition asteroids, for which the synthetic elements were not available. However, as of recently the synthetic proper elements are being computed also for the multi-opposition asteroids, thus the need for continuation of analytical computations is seriously questioned. On the other hand, the speed with which the analytical proper elements can be computed for a large number of asteroids speaks in their favor, thus the decision to cease their maintenance must be justified in terms of the accuracy required for their application. If indeed they are of generally so much poorer accuracy with respect to synthetic proper elements, as indicated on the basis of the above mentioned limited tests, and possibly also insufficient for the more and more demanding family classification applications, then their maintenance can safely be discontinued in the future.

For the purpose of determination of the accuracy of analytical proper elements, a comprehensive analysis has been envisaged and its realization recently commenced. It should consist of two different tests, one that exploits the plain comparison of the analytical and synthetic values, with the aim to determine whether they differ more then the corresponding error bars of synthetic elements permit, and the other, much more demanding but which should represent an ultimate test, consisting in repeating the above mentioned determination of instabilities of analytical elements themselves, just on a much larger sample of asteroids (on the order of 1 000 objects).

For the moment we can present here only some preliminary results of the comparison of analytical and synthetic proper elements, obtained for a limited sample of 10 000 numbered asteroids, but indicative of the method and results to be expected for the entire proper elements catalog with nearly 600 000 entries.

In Fig. 2 we show the distribution of differences for the three proper elements, and in Fig. 3 the same for the two frequencies.

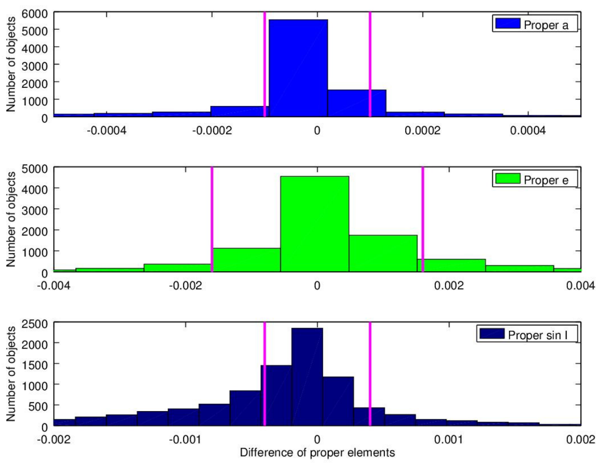

Fig. 2. *Distribution of differences (in the sense analytical – synthetic) of the proper semimajor axes (top), proper eccentricities (middle), and sine of proper inclination (bottom). The two vertical lines in each panel represent the corresponding mean standard deviation of synthetic proper elements.*

Even by a visual inspection of the three panels of Fig. 2 one finds that the results are as expected, the majority of differences in the case of semimajor axes and eccentricities are within the uncertainty of the synthetic elements, while for the sine of proper inclination the fraction of asteroids outside the uncertainty boundaries is larger, since the differences are in this case more spread and the mean standard deviation of synthetic proper $\sin I_p$ is really small, which gives rise to somewhat shallower distribution. Approximately 30% of the asteroids have difference of proper semimajor axes larger than the mean standard deviation of synthetic a_p, and the corresponding RMS of differences is equal to $1.1 \cdot 10^{-3}$ au. For the proper eccentricities these numbers are 25% and $6.4 \cdot 10^{-3}$, and 49% and $1.7 \cdot 10^{-3}$ for the sine of proper inclination. The conclusion is that the differences are for the most part not very big, in particular in comparison with the typical extensions of asteroid families in terms of proper elements (several hundredths to a few tenths) , but large enough to indicate the significantly lower accuracy of the analytical with respect to the synthetic proper elements.

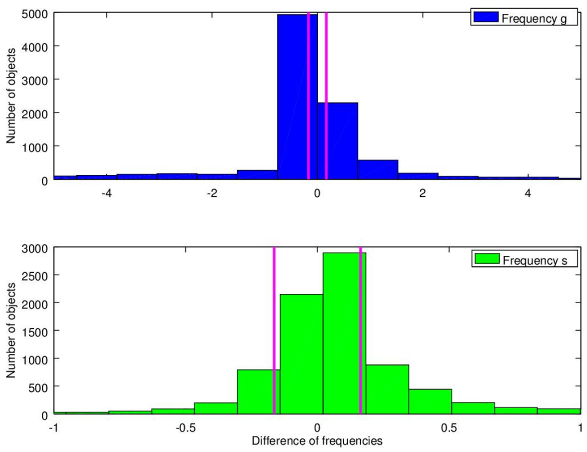

Fig. 3. *The same as Fig. 2, but for the rates of precession of perihelia (top), and of the nodes (bottom).*

As expected, the differences of the precession rates of proper angles are much larger, since the analytical values, in particular of the frequency g, are very sensitive to small divisors, giving rise to some unrealistic results in the vicinity of mean motion resonances. Hence, also the statistical estimates like those presented above for the proper elements do not have much meaning, but for the sake of completeness let us state that for the frequency g the fraction of differences larger than the mean uncertainty of synthetic values is 64% with the RMS of 9.8 arcsec/y (providing four anomalous values of more than 500 arcsec/y are excluded), and for frequency s these numbers are 49% and 3.9 arcsec/y, respectively.

Results of a slightly different kind of analysis are shown in Figs. 4 and 5. In Fig. 4 the differences of proper eccentricities and of the sines of proper inclinations are shown as functions of the synthetic proper sines of inclination and eccentricities themselves. This is done in such a way because it is well known that eccentricities more affect inclinations and vice versa. The points in the plot represent the mean values of proper element differences obtained by sorting the asteroids in terms of the argument (i.e. the corresponding proper element), dividing the sample in 20 bins of 500 objects each, and computing the individual bin averages in both coordinates. In Fig. 5 the values on both axes are combined in the "square root of the squares" sense: $\sqrt{(de)^2 + [d(\sin I)]^2}$ vs. $\sqrt{e^2 + \sin^2 I}$, such that the panels give the mean values for each bin (top) computed in the same way as in Fig. 4, and the corresponding standard deviations (bottom).

As already mentioned the analytical proper elements are computed from the series expansion of the disturbing function truncated at degree four in eccentricity and inclination, in the first order with respect to perturbing mass, and of degree two in the second order. Thus, one should expect that the accuracy of the analytical proper elements rapidly deteriorates with increasing eccentricity and inclination, and that this will also be reflected in differences with respect to the reference synthetic elements. It has been actually recommended by Knežević et al. (1995) to use the analytical proper elements only up to some $15^\circ - 17^\circ$ of inclination, and those computed by means of the semianalytical theory by Lemaitre and Morbidelli (1994) above.

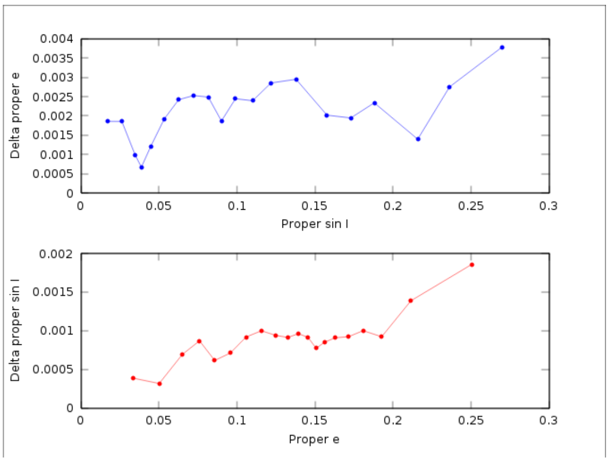

Fig. 4. *The differences of proper elements as functions of the elements themselves: Δe_p vs. $\sin I_p$ (top), $\Delta \sin I_p$ vs. e_p (bottom).*

The expected trend of increase of the spread of differences with increasing value of the considered proper element is visible in both panels of Fig. 4, but in the case of $(\sin I_p, \Delta e_p)$ plot (top), the general trend is not that obvious, rather it is revealed only through the somewhat smaller mean Δe_p values for $\sin I_p < 0.05$, and somewhat higher values for $\sin I_p > 0.25$; in the rest of the plot, the effect is practically not there. The $(e_p, \Delta \sin I_p)$ relation (bottom), on the other hand, exhibits nearly monotonous increase, but with the total change reaching only a half of the other one, in agreement with the similar ratio of spreads of differences observed in the middle and bottom panels of Fig. 2.

If both proper elements and the corresponding differences of analytical and synthetic values are combined in the "square root of the squares" sense (Fig. 5), the expected trend of increase of the instabilities of analytical elements due to the growth of the size of neglected terms in the expansion of the disturbing function is clearly observable for the mean values. The corresponding standard deviations exhibit a somewhat less pronounced general trend of increase, also reaching much larger values. The two results, one on the mostly moderate average differences of proper values, and the other on the significantly larger standard deviations, taken together, confirm the conclusion drawn on the basis of Fig. 2, that indeed there are many analytical proper elements which are fairly similar to their synthetic counterparts, but that there are also quite a lot of those with anomalously big deviations giving rise to a large, and growing spread.

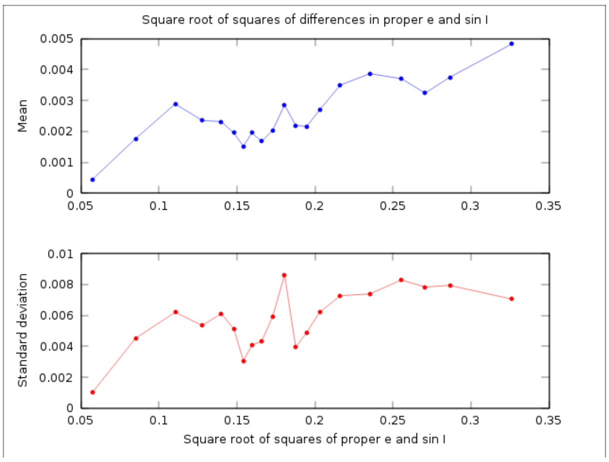

Fig. 5. *The combined differences of proper elements as function of the combined values of the elements themselves. Mean values (top), standard deviations (bottom).*

All this means that the accuracy of analytical proper elements and trends of its variation may in some cases be more complicated than one would expect, but this is the problem that requires more complete and thorough analysis then the simple attempt presented here.

4. CONCLUSIONS

Although the above described advances perhaps do not represent the truly decisive breakthroughs in computation and stability assessment of the asteroid proper elements, they can still be considered important improvements offering solutions to some of the practical problems encountered in the past.

While the problem of exact computation of the perihelion frequencies for nearly circular orbits is technical in essence and affects a rather limited sample of asteroids, the other two problems are of more general significance and affecting much larger populations of objects.

The advantage of computation of secular resonant proper elements is in their superior accuracy in the specific case of resonant orbits in comparison with the usual synthetic elements, the drawbacks that they have to be adjusted to each considered resonance in turn, and that they cannot be combined with synthetic elements to identify families spilling over the secular resonance separatrices; even using the sole resonant proper elements for resonant families classification requires the standard HCM metrics (Zappalà et al. 1990) and velocity cut-offs to be adjusted too.

In spite of being quite preliminary, the results of the stability tests of analytical proper elements clearly indicate that significant fraction of these elements exhibit large instabilities and that they are of generally inferior accuracy with respect to their synthetic counterparts. Still, to draw final conclusions we must wait for more comprehensive and complete assessment. Although in general in agreement with expectations, a slightly irregular behavior of the dependence of the differences of analytical and synthetic proper elements on the elements themselves in the sample of asteroids analyzed here, deserves to be investigated in more detail.

Acknowledgements – Andrea Milani's reading of the manuscript and suggesting the improvements is gratefully acknowledged. This work has been completed in the framework of the project F-187 "Dynamics of the solar system bodies" of the Serbian Academy of Sciences and Arts.

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ОДРЕЂИВАЊЕ СОПСТВЕНИХ ЕЛЕМЕНАТА АСТЕРОИДА: НЕКА НЕДАВНА ПОБОЉШАЊА

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УДК 523.44-323.8
Прегледни рад по позиву

У раду су кратко представљена нека недавна побољшања одређивања сопствених елемената кретања астероида. Иако се не ради о великим продорима у рачунању и оцени стабилности сопствених елемената, ова побољшања се ипак могу сматрати важним јер су допринела решавању неких проблема с којима смо се сусретали у прошлости. Проблем нетачног одређивања фреквенција прецесије перихела за путање веома малих ексцентричности решен је рачунањем фреквенција помоћу фреквенционо модификоване Фуријеове трансформације. Синтетички резонантни сопствени елементи специјално прилагођени датој резонанци помогли су да се докаже постојање астероидне фамилије Astraea. Прелиминарна оцена стабилности сопствених елемената добијених помоћу аналитичке теорије указује на њихове слабије перформансе у поређењу са одговарајућим синтетичким елементима, што говори у прилог обустављању њиховог даљег редовног рачунања; коначну одлуку ипак треба донети на основу свеобухватније и поузданије директне процене њиховог појединачног и просечног узорачког одступања од константности.